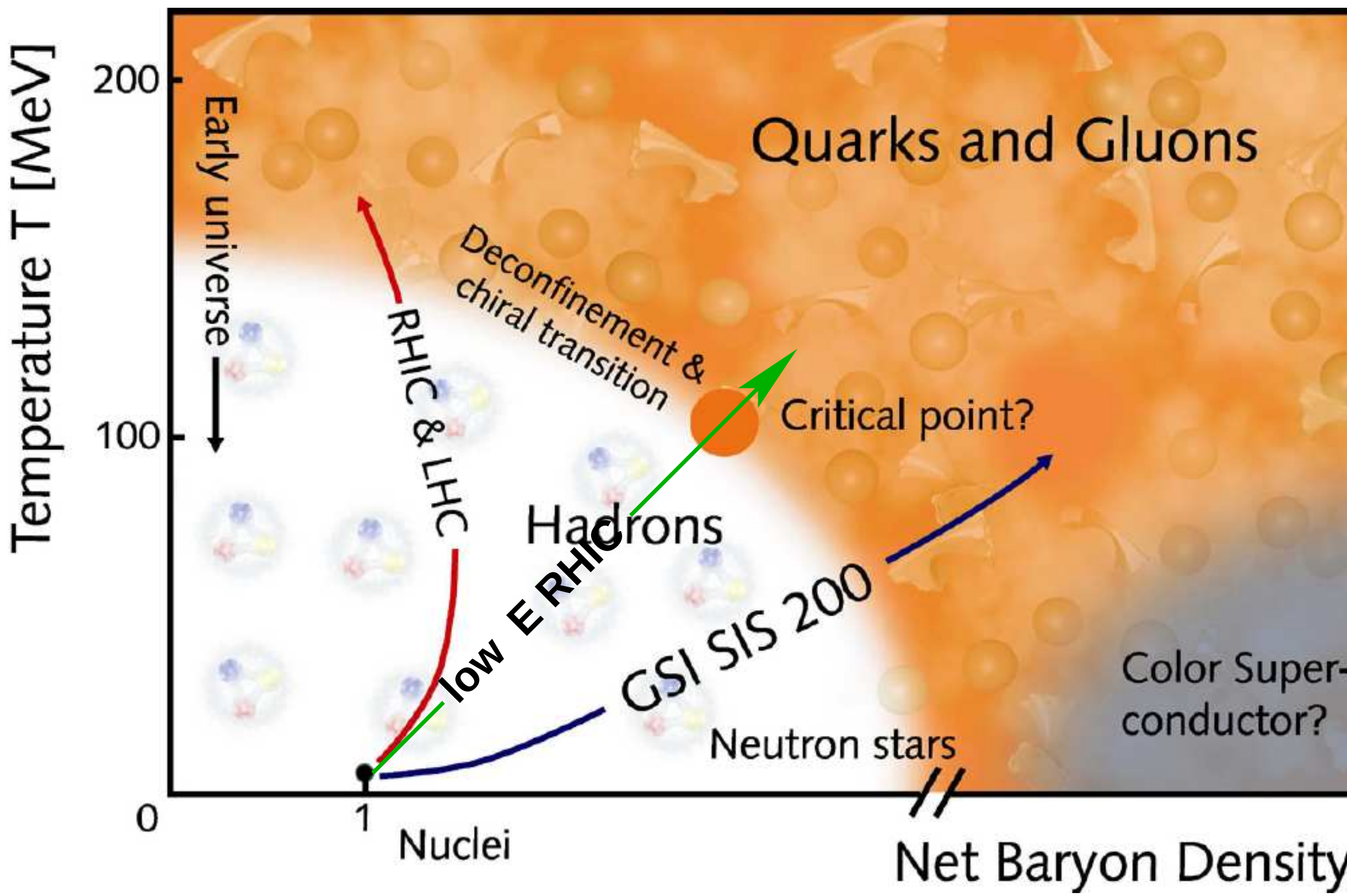
Lattice results on the QCD critical point

Frithjof Karsch, BNL & Bielefeld University



Lattice results on the QCD critical point

- Where is the critical point?

Bulk thermodynamics with non-vanishing chemical potential

$$Z(V, T, \mu) = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\, \mathrm{e}^{-S_E(V,T,\mu)}$$
$$= \int \mathcal{D}\mathcal{A}\, \left[det\ M(\mu)\right]^f\, \mathrm{e}^{-S_G(V,T)}$$

$\Uparrow$complex fermion determinant;

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$$= \int \mathcal{D}\mathcal{A}\,\left[det\; M(\mu)\right]^f\; \mathrm{e}^{-S_G(V,T)}$$

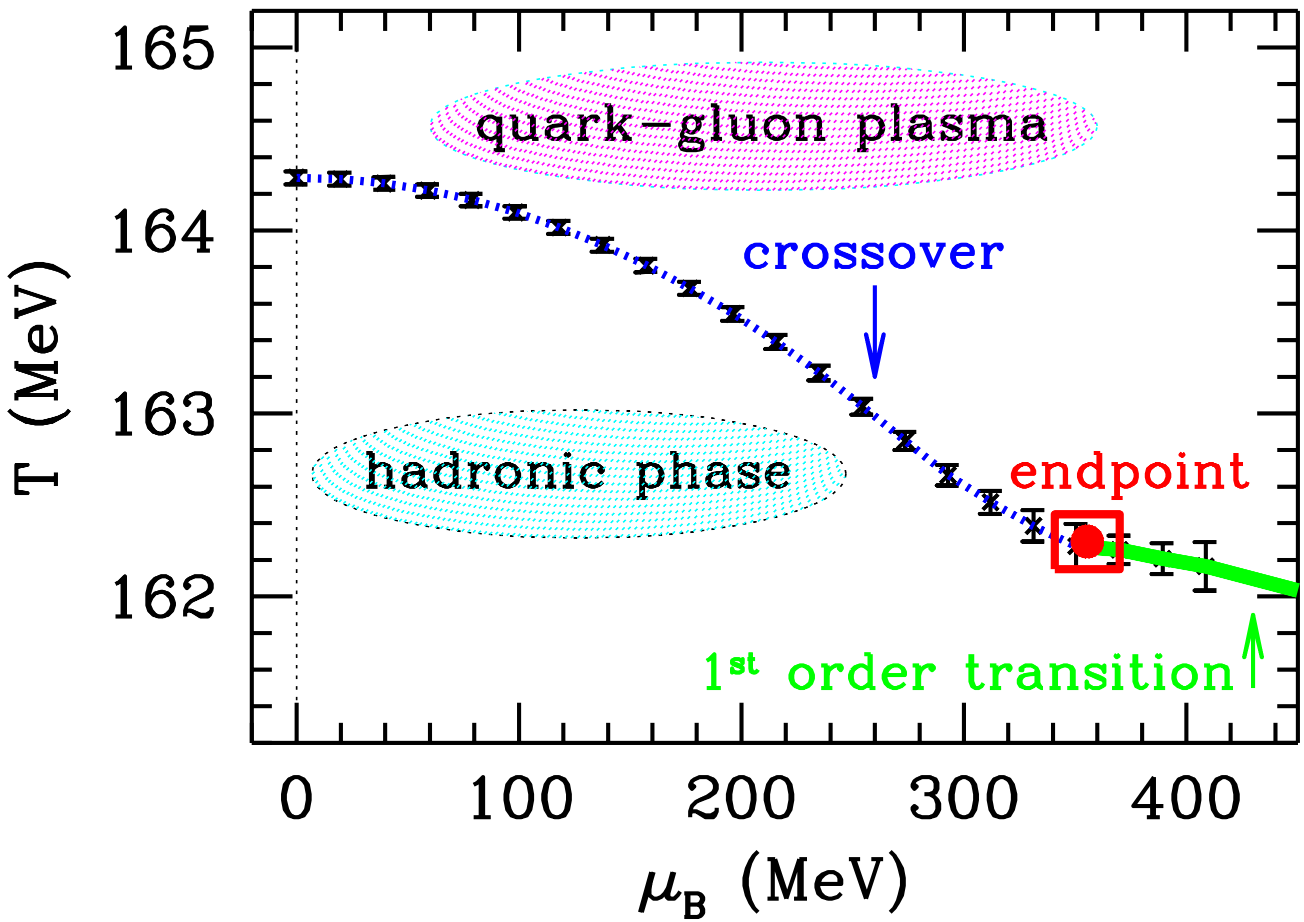
⇑complex fermion determinant;

ways to circumvent this problem:

- reweighting: works well on small lattices; requires exact evaluation of $detM$
  Z. Fodor, S.D. Katz, JHEP 0203 (2002) 014
- Taylor expansion around $\mu = 0$: works well for small μ;
  C. R. Allton et al. (Bielefeld-Swansea), Phys. Rev. D66 (2002) 074507
  R.V. Gavai, S. Gupta, Phys. Rev. D68 (2003) 034506
- imaginary chemical potential: works well for small μ; requires analytic continuation
  Ph. deForcrand, O. Philipsen, Nucl. Phys. B642 (2002) 290
  M. D'Elia and M.P. Lombardo, Phys. Rev. D64 (2003) 014505
- canonical ensemble: need to evaluate fermion determinant
  K.-F. Liu, Int. J. Mod. Phys. B16 (2002) 2017
  S. Kratochvila and P. de Forcrand, PoS LAT2005, 167 (2006)

Early lattice results

reweighting:



$(6-12)^3 \times 4$

Z. Fodor, S. Katz, JHEP 0404 (2004) 050

Reweighting and Lee-Yang zeroes

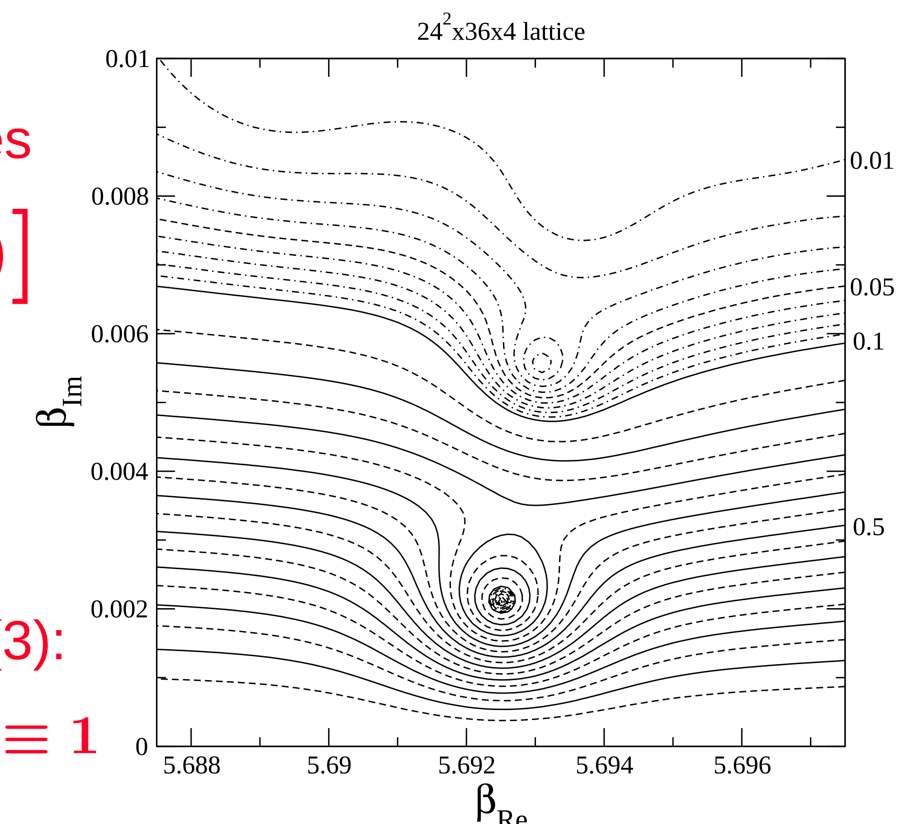
$$\mathcal{Z}_{\mathrm{norm}}(\beta_{\mathrm{Re}},\beta_{\mathrm{Im}},\mu) \equiv \left|\frac{\mathcal{Z}(\beta_{\mathrm{Re}},\beta_{\mathrm{Im}},\mu)}{\mathcal{Z}(\beta_{\mathrm{Re}},0,0)}\right|$$

$$= \left|\left\langle e^{6i\beta_{\mathrm{Im}}N_{\mathrm{site}}\Delta P} e^{i\theta} \left| e^{(N_{\mathrm{f}}/4)(\ln\det M(\mu)-\ln\det M(0))}\right|\right\rangle_{(\beta_{\mathrm{Re}},0,0)}\right|$$

additional complication in QCD

phase of fermion matrix contributes

$$e^{i\theta} \sim e^{i\left[c\mu V + \mathcal{O}(\mu^3 V)\right]}$$



the sign problem severely limits the range of applicability of the reweighting method

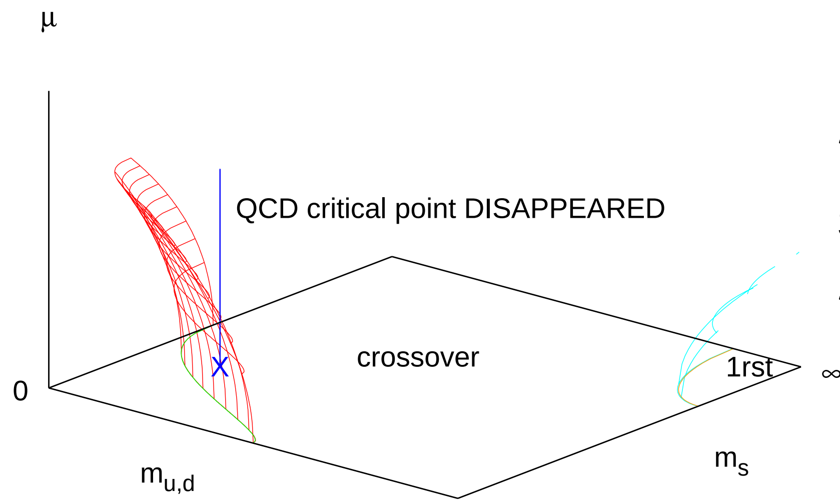
S. Ejiri, PR D 73 (2006) 054502

SU(3): $e^{i\theta} \equiv 1$

Status 2008: ???

- the devils advocats

  imaginary μ: Ph. de Forcrand, O. Philipsen, NP B642 (2002) 290; updated at Lattice 08 and xQCD 08



$n_f = 3,\ N_\tau = 4$:
standard staggered
$m_c(\mu) = m_c(0) - A\mu^2$
$\Uparrow$

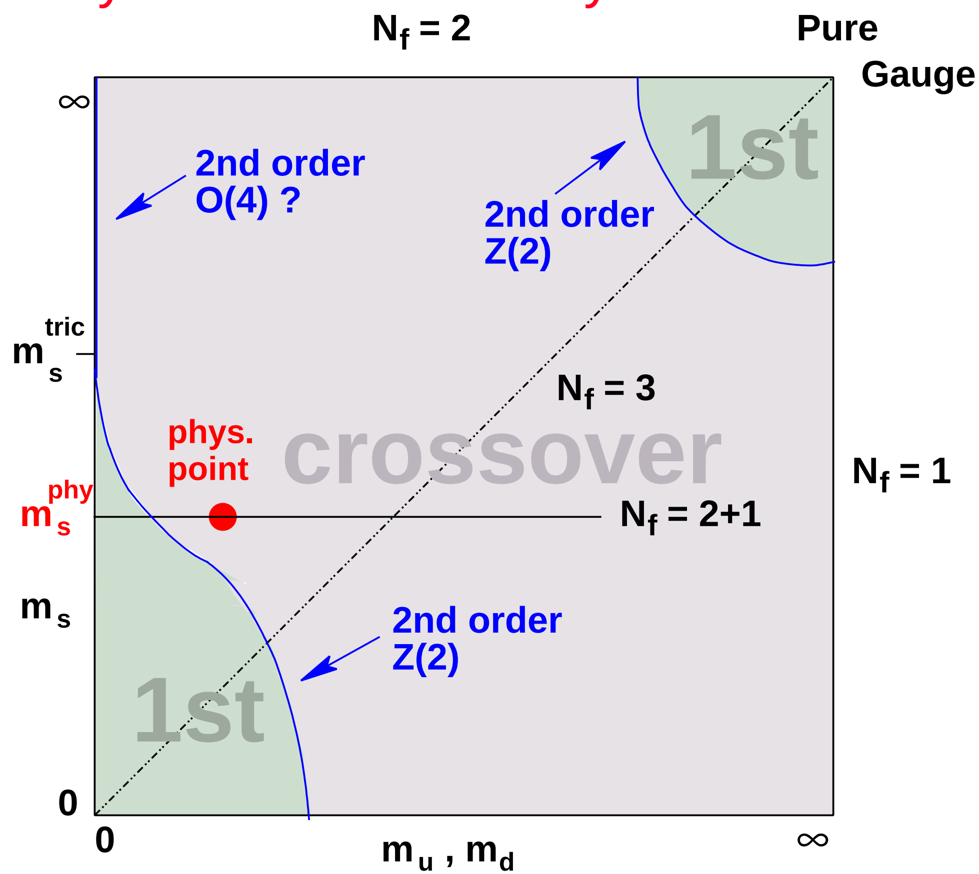
- the convinced

  Taylor expansion: R. Gavai, S. Gupta, PR D71 (2005) 114014; arXiv:0806.2233

N_τ	$T_c(\mu_c)/T_c(0)$	μ_c/T	comment
4	0.95(2)	1.1(2)	2005: 2-flavor; standard staggered
6	0.94(1)	1.8(1)	2008: 2-flavor; no $V \to \infty$
4	0.99(2)	2.2(2)	Fodor+Katz: (2+1)-flavor, standard staggered

Phase diagram for $\mu_B = 0$

- already the $\mu_B = 0$ phase diagram is not fully explored
- phase boundary is known to be very sensitive to cut-off effects



- $N_\tau = 4$, standard staggered fermions (the setup used by deF+P):
  $\Rightarrow m_{ps}^{crit} \simeq 300$ MeV for $n_f = 3$, *i.e.* larger than physical m_π

FK, E. Laermann, C Schmidt, PL B520 (2001) 41

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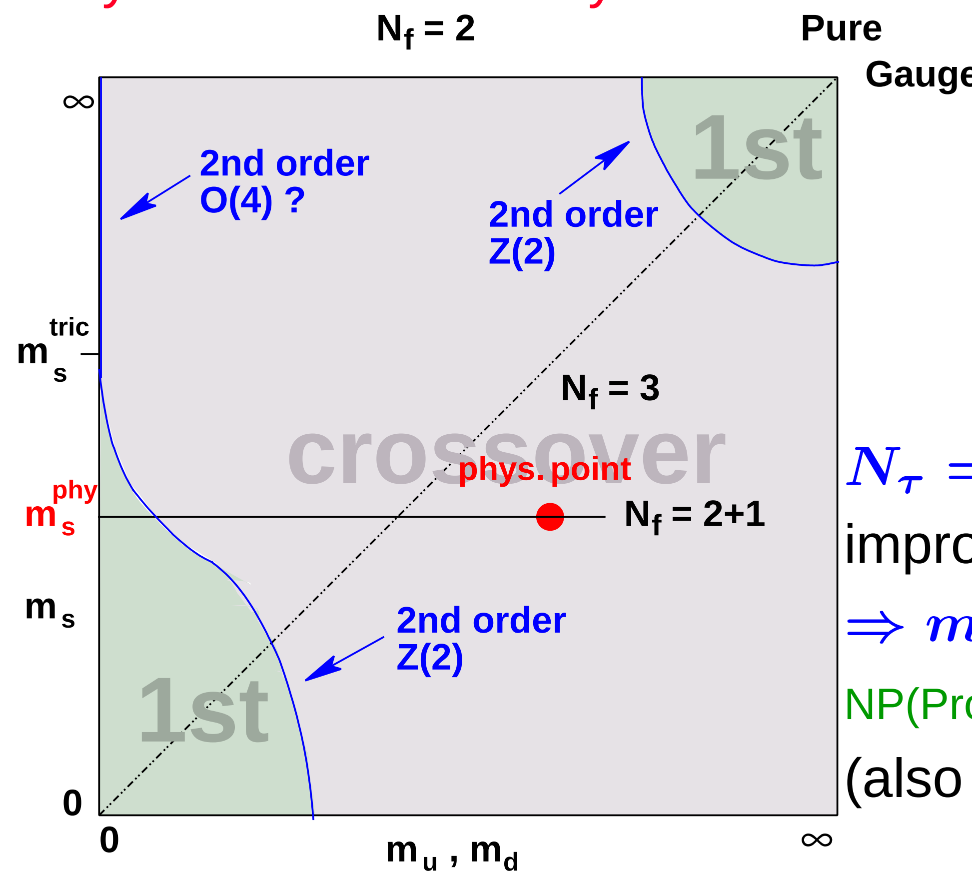

$N_\tau = 4$
improved action
$\Rightarrow m_{ps}^{crit} \simeq 70$ MeV
NP(Proc.Suppl) 129 (2004) 614
(also $N_\tau = 6$, unimp.)

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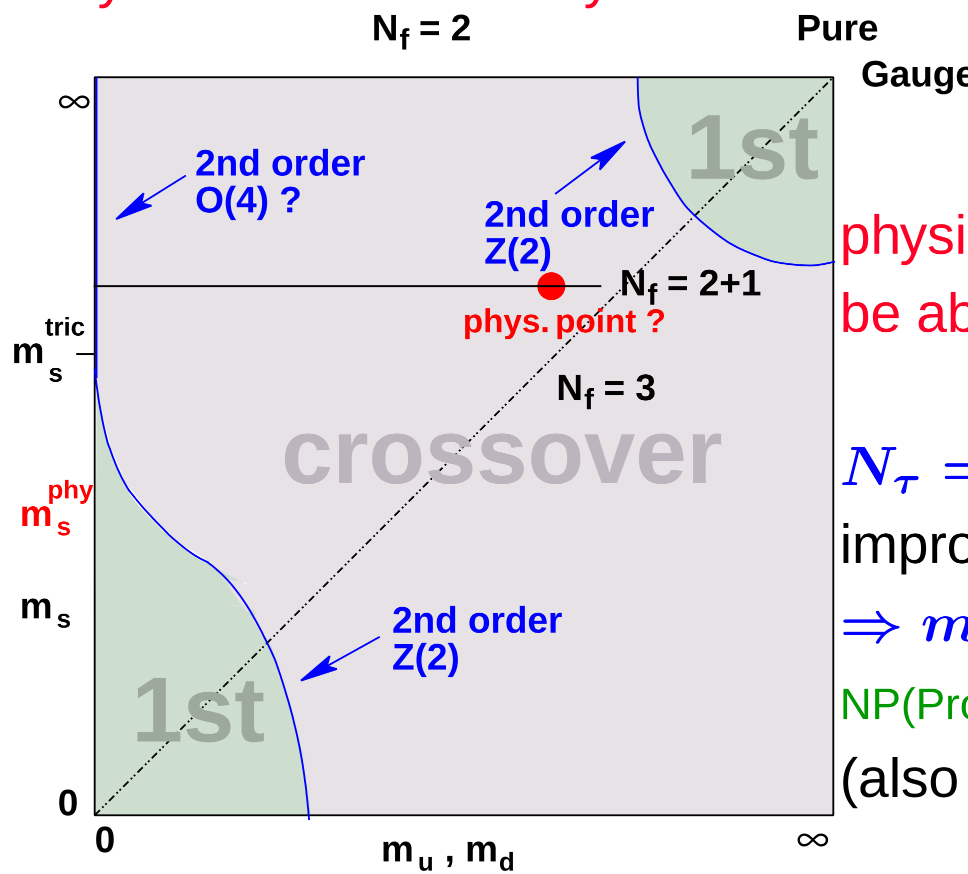

physical point may be above m_s^{tri}

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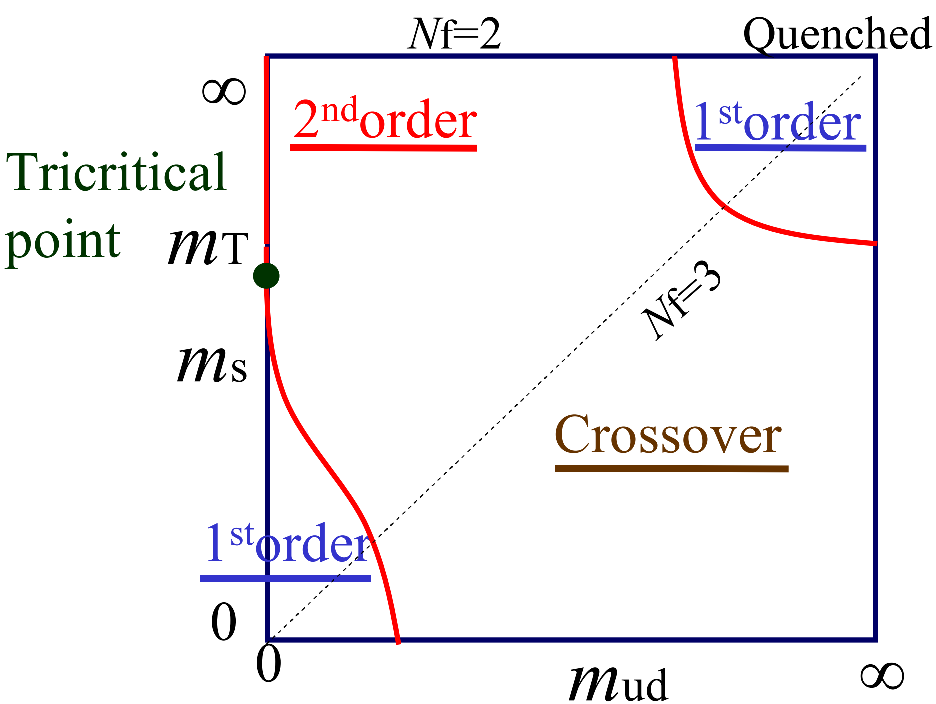
Mean field argument

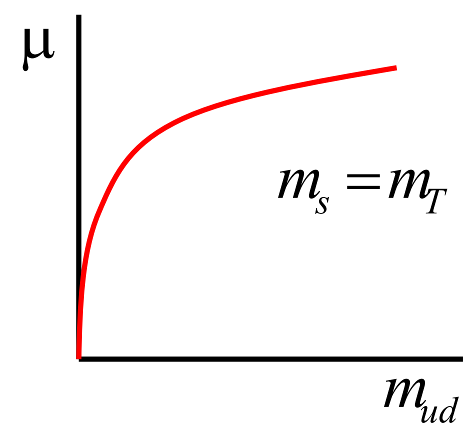
- Sigma model prediction near tri-critical point on the ms axis.

$$V_{\text{eff}}(\sigma) = \frac{1}{2}a\sigma^2 + \frac{1}{4}b\sigma^4 + \frac{1}{6}c\sigma^6 - h\sigma$$

Critical point: $\dfrac{\partial^n V_{\text{eff}}}{\partial \sigma^n} = 0 \quad (n = 1,2,3)$



$$b \sim (m_T - m_s) \quad \Rightarrow \quad b \sim \mu^2$$



$$m_{ud}^{\text{crit}} \sim (m_T - m_s)^{5/2} \quad \Rightarrow \quad m_{ud}^{\text{crit}} \sim \mu^5$$

Bulk thermodynamics for small μ_q/T

- Taylor expansion of pressure up to $\mathcal{O}\left((\mu_q/T)^6\right)$

$$\frac{p}{T^4} = \frac{1}{VT^3}\ln \mathcal{Z} = \sum_{n=0}^{\infty} c_n(T)\left(\frac{\mu_q}{T}\right)^n \simeq c_0 + c_2\left(\frac{\mu_q}{T}\right)^2 + c_4\left(\frac{\mu_q}{T}\right)^4 + c_6\left(\frac{\mu_q}{T}\right)^6$$

quark number density
$$\frac{n_q}{T^3} = 2c_2\frac{\mu_q}{T} + 4c_4\left(\frac{\mu_q}{T}\right)^3 + 6c_6\left(\frac{\mu_q}{T}\right)^5$$

quark number susceptibility
$$\frac{\chi_q}{T^2} = 2c_2 + 12c_4\left(\frac{\mu_q}{T}\right)^2 + 30c_6\left(\frac{\mu_q}{T}\right)^4$$

an estimator for the radius of convergence

$$\left(\frac{\mu_q}{T}\right)_{crit} = \lim_{n\to\infty}\left|\frac{c_{2n}}{c_{2n+2}}\right|^{1/2}$$

$c_n > 0$ for all n
$\Rightarrow$ singularity for real μ

Estimating $T_c(\mu_c)$ and μ_c/T

How can we estimate the location of the critical endpoint from expansion coefficients of the Taylor series?

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- The approach taken by Gavai and Gupta: arXiv:0806.2233

  ..As one comes down in T from large values of T, the series coefficients remain positive down to some lowest value of T/T_c....At this temperature the radius of convergence is...

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a potential problem with unimproved staggered fermions:

unimproved staggered fermions have μ-dependent $\mathcal{O}(a^2)$ corrections that are positive and significant even in the ideal gas limit.
For instance, they generate a positive $\mathcal{O}(\mu^6)$ term

P. Hegde et al, Eur.Phys. J. C55 (2008) 423

$$\mathcal{O}(a^2) \sim \left(1 + \frac{147}{31}\left(\frac{\mu}{\pi T}\right)^2 + \frac{105}{31}\left(\frac{\mu}{\pi T}\right)^4 + \frac{21}{31}\left(\frac{\mu}{\pi T}\right)^6\right)\frac{1}{N_\tau^2}$$

Bulk thermodynamics for small μ_q/T

- Taylor expansion of pressure up to $\mathcal{O}\left((\mu_q/T)^6\right)$

  $N_\tau = 4$, $\mathcal{O}(a^2)$ improved staggered fermions, $m_\pi \simeq 770$ MeV

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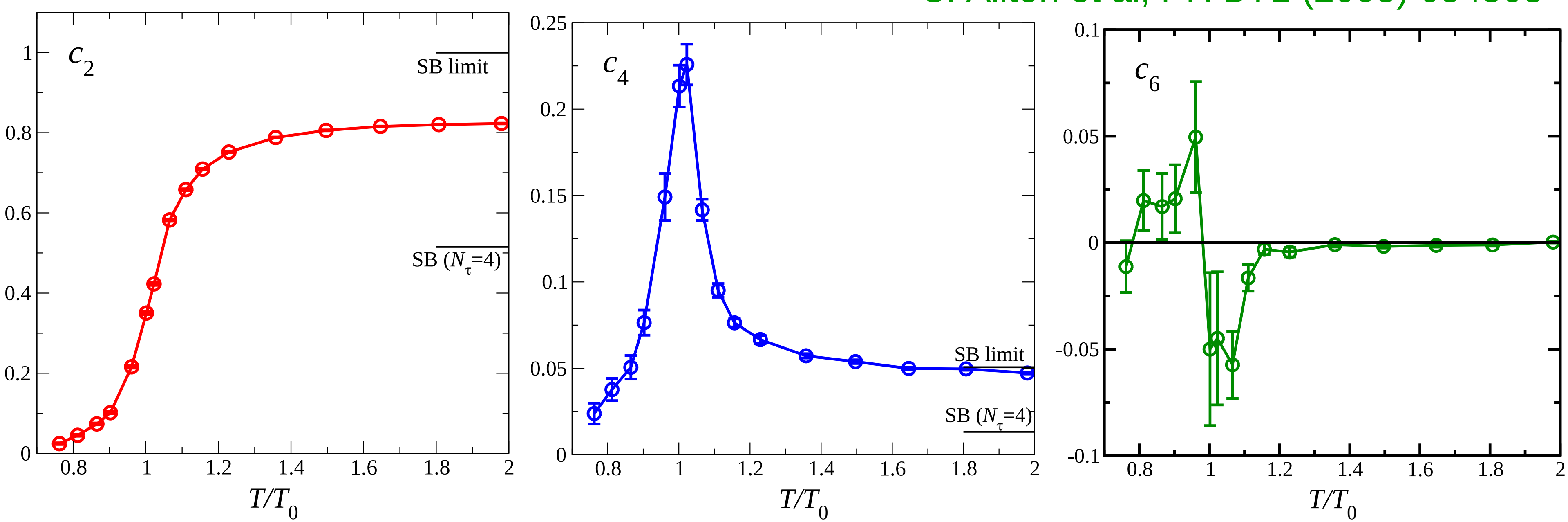
C. Allton et al, PR D71 (2005) 054508



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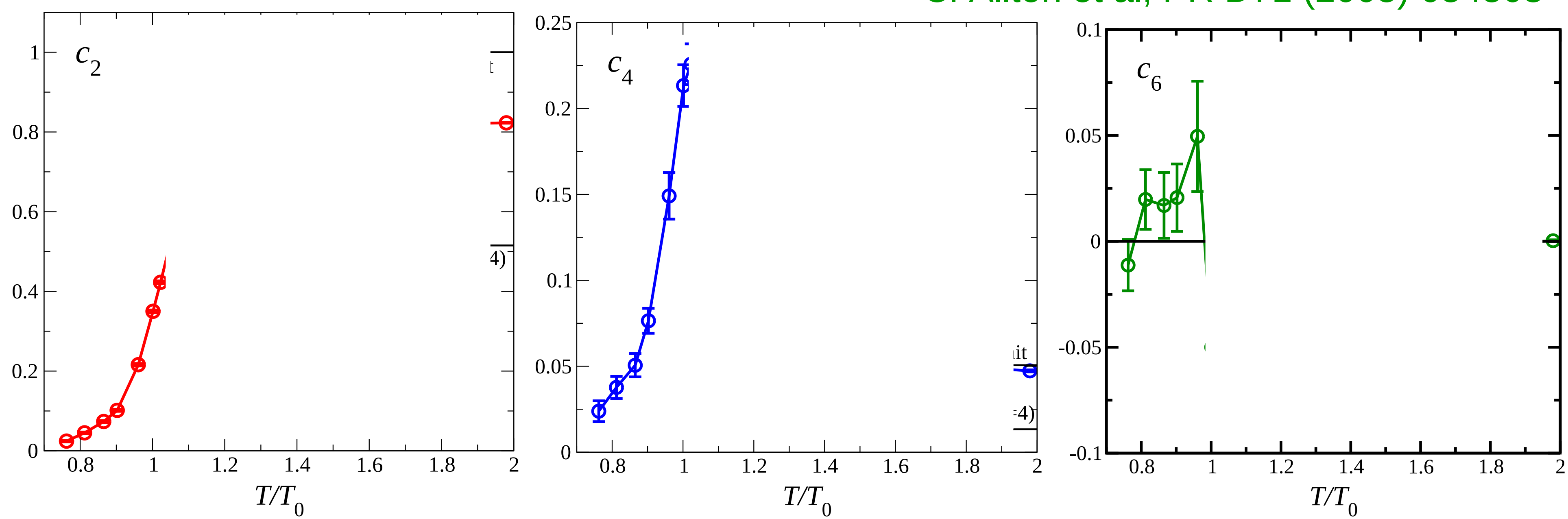

$c_n > 0$ for all n and $T \lesssim 0.95\, T_c$ $\Leftrightarrow$ singularity for real μ (positive μ^2)

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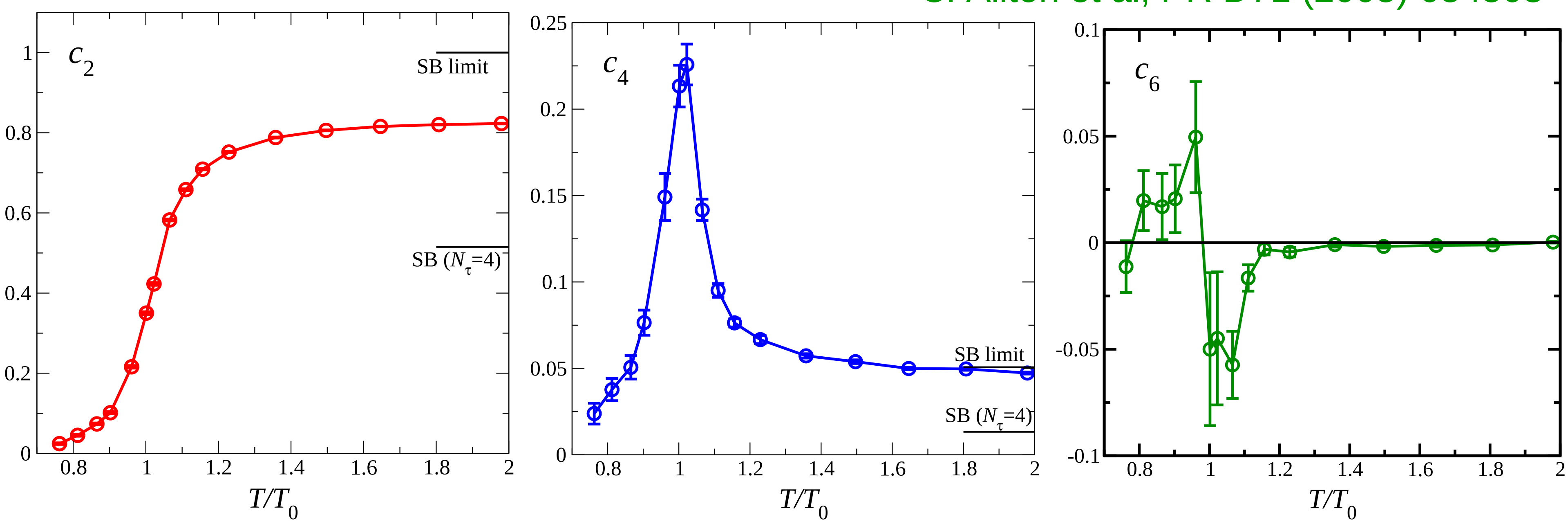
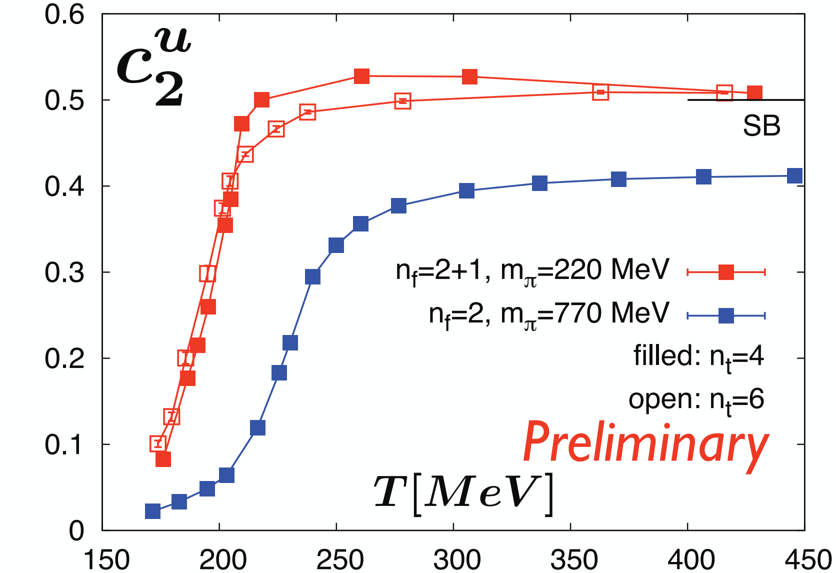
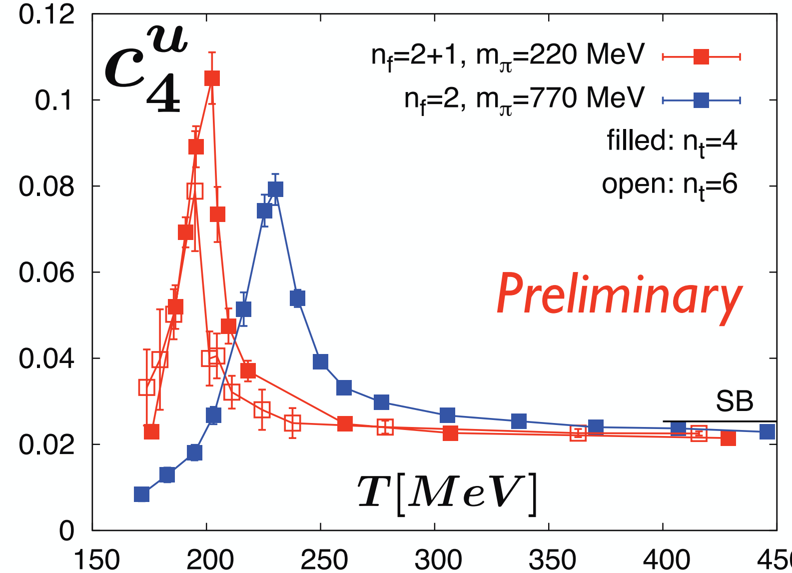


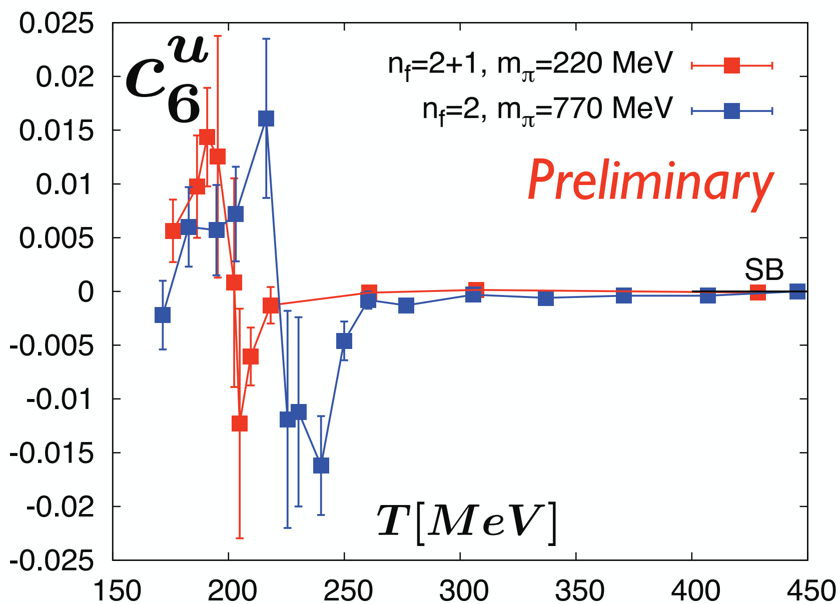
irregular sign of c_n for $T \gtrsim T_c$ $\Leftrightarrow$ singularity in complex plane

Taylor expansion of the pressure

- **Results for expansion coefficients** $c_{i,j,k}^{u,d,s}$







Cut-off dependance:

→ **Small cut-off effects in the transition region (similar to p, e-3p, ...)**

Mass dependance:

→ **Tc decreases with decreasing mass**

→ **Fluctuations increase with decreasing mass**

red: RBC-Bielefeld, preliminary
blue: PRD71:054508,2005.

Hadronic fluctuations at $\mu_q = 0$

- expect 2^{nd} order transition in 3-d, O(4) symmetry class;

  scaling field: $t = \left|\frac{T-T_c}{T_c}\right| + A\left(\frac{\mu_q}{T_c}\right)^2$, $\mu_{crit} = 0$

  singular part: $f_s(T,\mu_u,\mu_d) = b^{-1} f_s(tb^{1/(2-\alpha)}) \sim t^{2-\alpha}$

$$c_2 \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial \mu_q^2} \sim t^{1-\alpha} \quad , \quad c_4 \sim \frac{\partial^4 \ln \mathcal{Z}}{\partial \mu_q^4} \sim t^{-\alpha} \qquad (\mu = 0)$$

- O(4)/O(2): $\alpha < 0$, small $\Rightarrow$

  $c_2 \sim \langle(\delta N_q)^2\rangle$ dominated by T-dependence of regular part

  $c_4 \sim \langle(\delta N_q)^4\rangle - 3\langle(\delta N_q)^2\rangle^2$ develops a cusp

  Y. Hatta, T. Ikeda, PRD67 (2003) 014028

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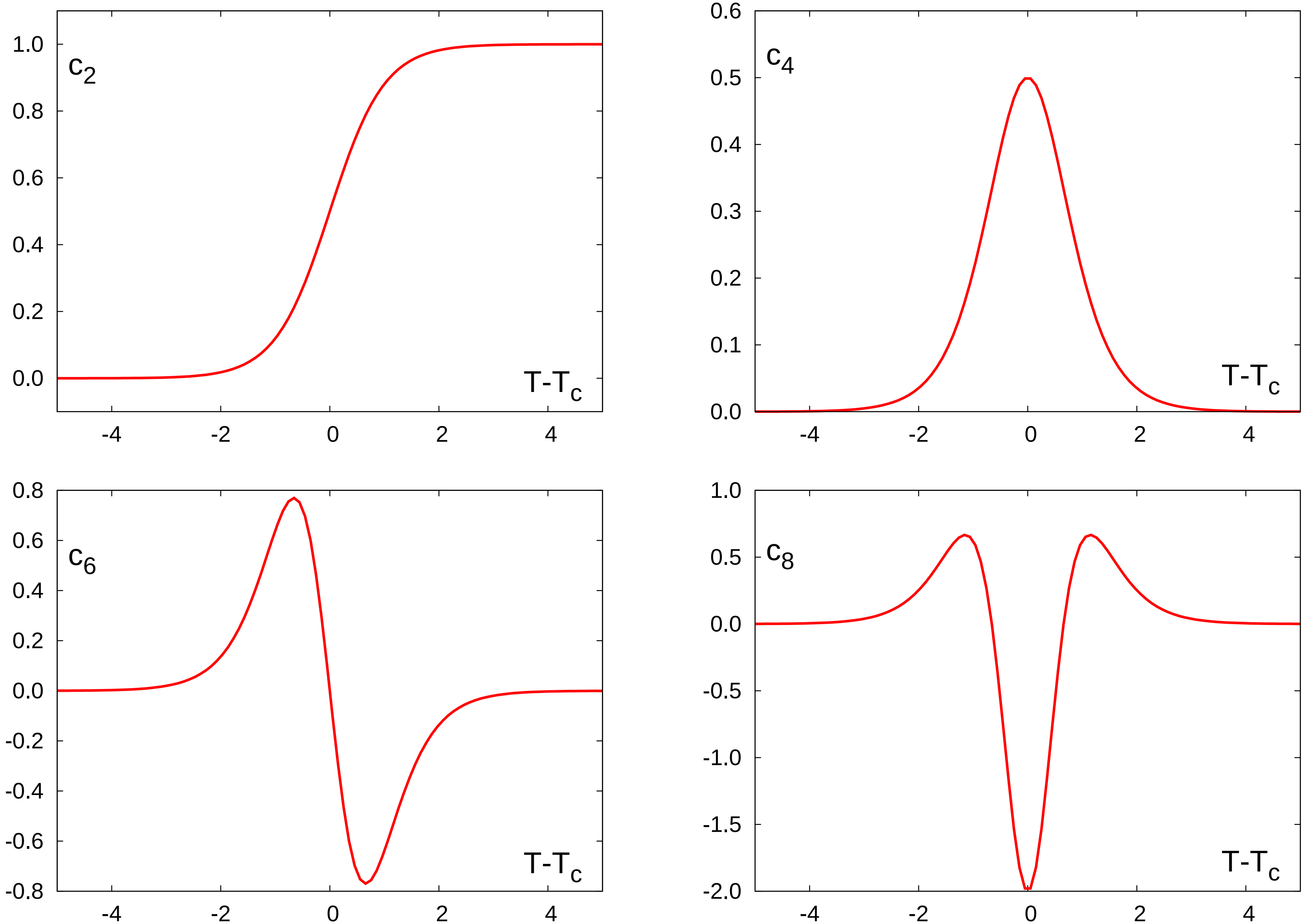
$$\epsilon \sim \frac{\partial \ln \mathcal{Z}}{\partial T} \sim t^{1-\alpha} \quad , \quad C_V \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial T^2} \sim t^{-\alpha} \quad (\mu = 0)$$

$\Rightarrow 2^{nd}$ derivative w.r.t μ_q "looks like energy density"
$\Rightarrow 4^{th}$ derivative w.r.t μ_q "looks like specific heat"

Generic expansion coefficients



similar in PNJL model: S. Roessner et al, PR D75 (2007) 034007

Estimating $T_c(\mu_c)$ and μ_c/T

How can we estimate the location of the critical endpoint from expansion coefficients of the Taylor series?

- need $c_n(T) > 0$ to have a singularity on the real axis
- expect hadron resonance gas to be a good approximation at low T:

  $c_n^{HRG} > 0$ for all n, but $r_n^{HRG} = \sqrt{1/(n+2)/(n+1)} \rightarrow 0$

$\Rightarrow$ conjecture:

the position of the first maximum of $c_n(T)$, e.g. at $T_n < T_c(0)$, gives an upper bound on $T_c(\mu_c)$ as one will find $c_{n+2}(T) < 0$ for $T > T_n$

- This is completely opposite to the criterium formulated by Gavai and Gupta

Hadronic fluctuations and the QCD critical point 9

- **Consequences for the phase diagram:**
  the radius of convergence

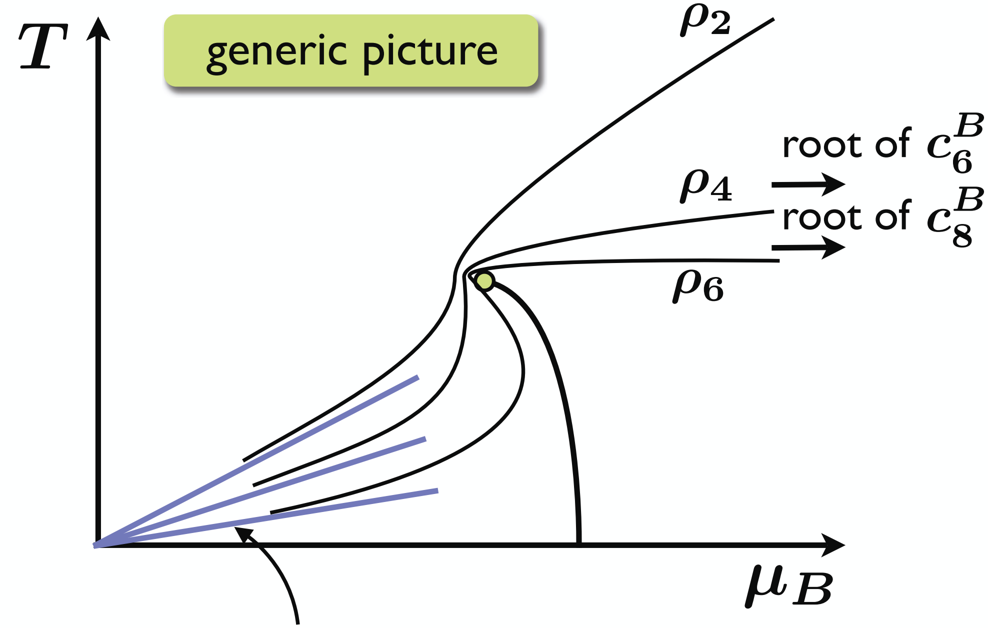
The radius of convergence can be estimated from the Taylor coefficients of the pressure:

$$\rho = \lim_{n\to\infty} \rho_n$$

with

$$\rho_n = \sqrt{\frac{c_n^B}{c_{n+2}^B}}$$

- for $T > T_c,\quad \rho_n \to \infty$
- for $T < T_c,\ \rho_n$ is bound by the transition line



The Resonance gas limit:

$$\frac{p}{T^4} = G(T) + F(T)\cosh\left(\frac{\mu_B}{T}\right)$$

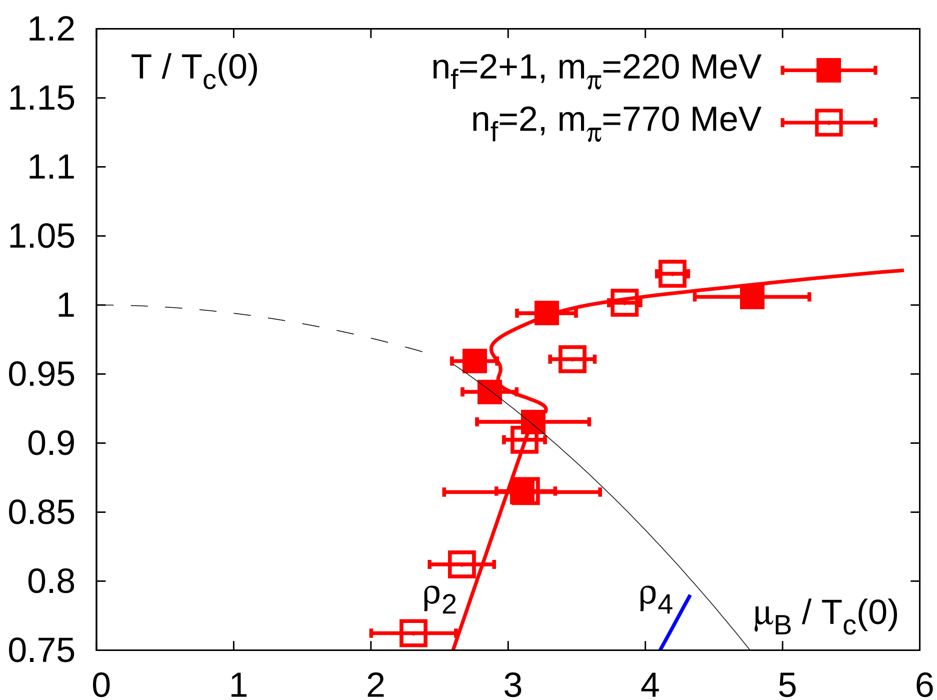
$$\longrightarrow \rho_n = \sqrt{1/(n+2)(n+1)}$$

$\longrightarrow$ **look for non-monotonic behavior in the radius of convergence**

Estimating $T_c(\mu_c)$ and μ_c/T

Status of the RBC-BI project

- calculations for $N_\tau = 4$ and 6; $N_\sigma = 4N_\tau$
- uses an $\mathcal{O}(a^2)$ improved staggered action (p4fat3)
- estimator for μ_c: $$\left(\frac{\mu_c(T)}{T_c(0)}\right)_n \equiv \rho_n = \frac{T}{T_c(0)}\sqrt{\frac{c_n}{c_{n+2}}}$$



- slight quark mass dependence

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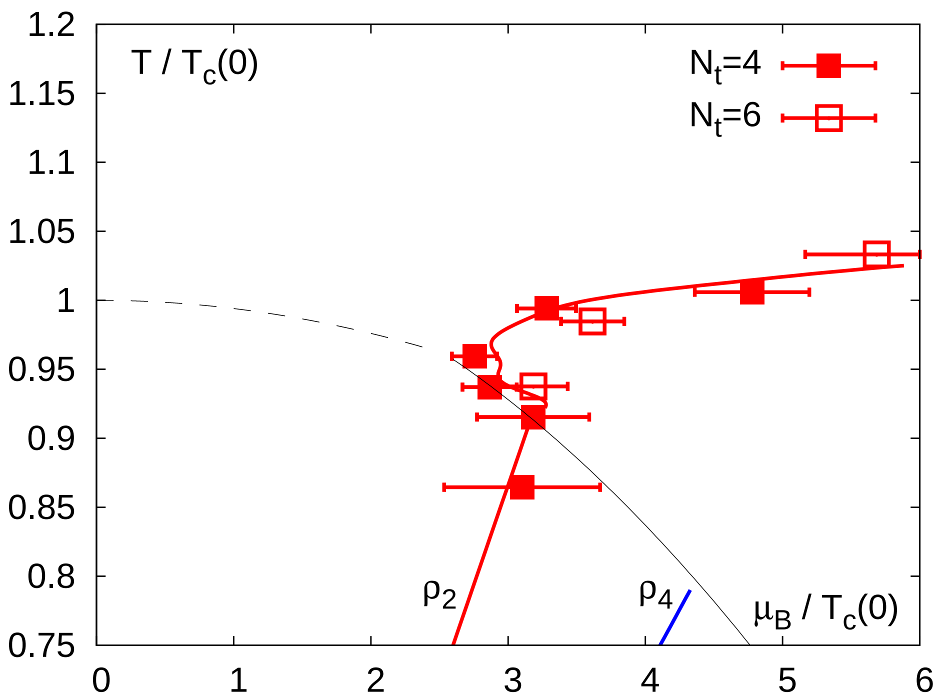

- slight quark mass dependence
- weak cut-off dependence

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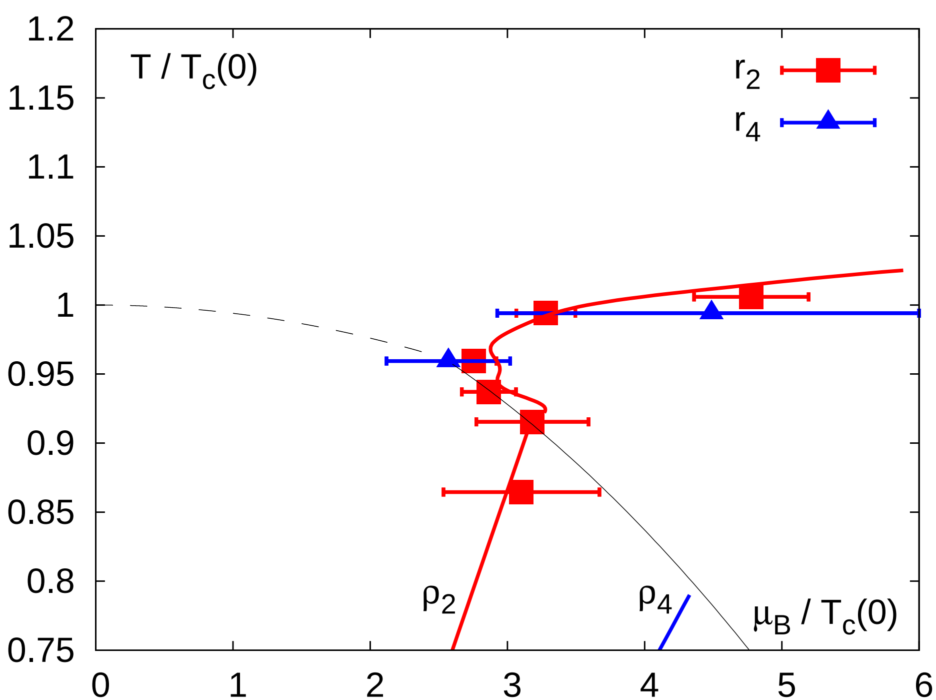

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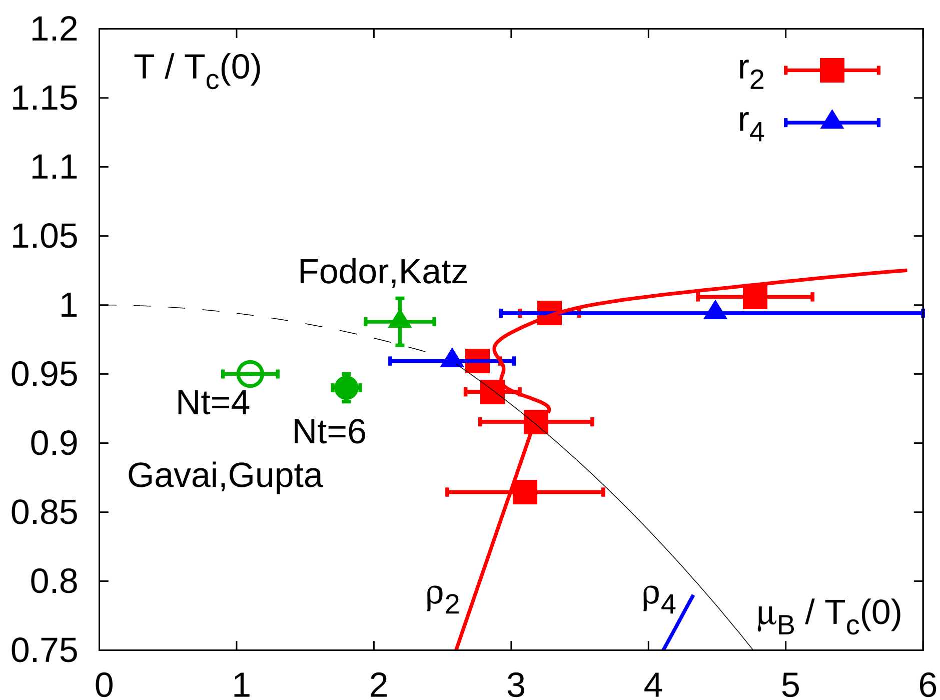

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Hadronic fluctuations at $\mu > 0$ from Taylor expansion coefficients at $\mu = 0$

$n_f = 2,\ m_\pi \simeq 770$ MeV: S. Ejiri, FK, K.Redlich, PLB633 (2006) 275
$n_f = 2+1,\ m_\pi \simeq 220$ MeV: RBC-Bielefeld, preliminary

- Taylor expansion of bulk thermodynamics in terms of $\mu_{u,d,s}$

$$\begin{aligned} \frac{p}{T^4} &\equiv \frac{1}{VT^3} \ln Z(V,T,\mu_u,\mu_d,\mu_s) \\ &= \sum_{i,j,k} c_{i,j,k} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_u}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k \end{aligned}$$

- expansion coefficients evaluated at $\mu_{u,d,s} = 0$ are related to fluctuations of $B,\ S,\ Q$ at $\mu_{B,S,Q} = 0$:

$\Uparrow$ baryon number, strangeness, charge fluctuations

event-by-event fluctuations at RHIC and LHC

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$n_f = 2+1,\ m_\pi \simeq 220$ MeV: RBC-Bielefeld, preliminary

- quadratic and quartic fluctuations

$$\chi_2^x = \frac{\partial^2 p/T^4}{\partial(\mu_x/T)^2} = \frac{1}{VT^3}\langle(\delta N_x)^2\rangle_{\mu=0} = \frac{1}{VT^3}\langle N_x^2\rangle_{\mu=0}$$

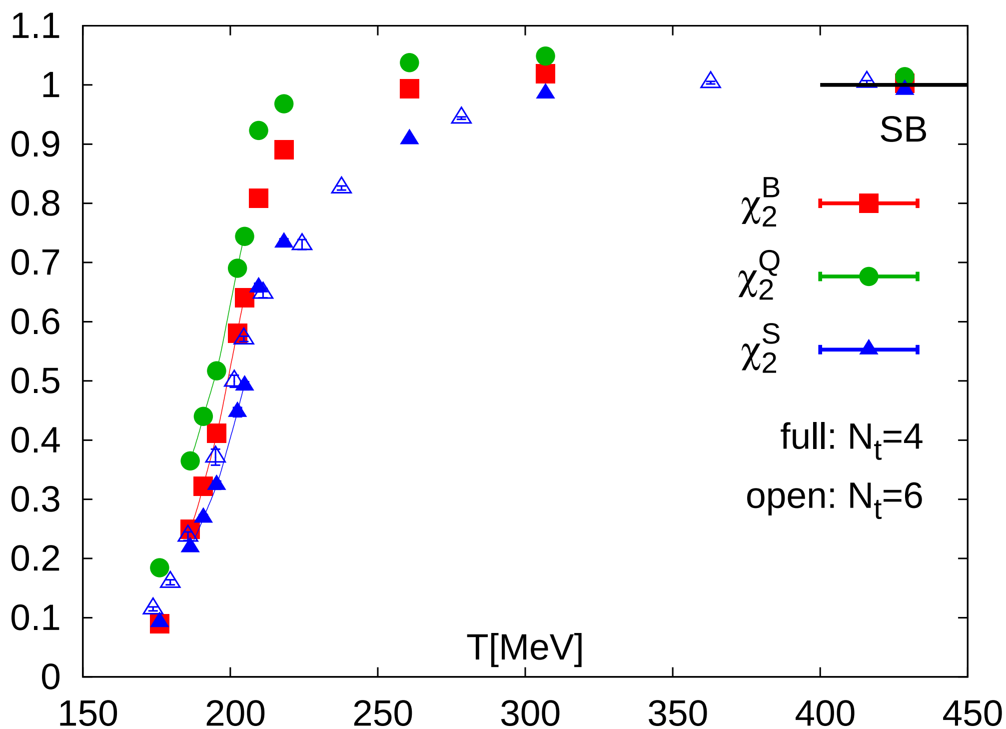
$$\begin{aligned}\chi_4^x &= \frac{\partial^4 p/T^4}{\partial(\mu_x/T)^4} = \frac{1}{VT^3}\left(\langle(\delta N_x)^4\rangle - 3\langle(\delta N_x)^2\rangle^2\right)_{\mu=0} \\ &= \frac{1}{VT^3}\left(\langle N_x^4\rangle - 3\langle N_x^2\rangle^2\right)_{\mu=0}\end{aligned}$$

with $x = u,\ d,\ s$ or $B,\ Q,\ S$

Quadratic fluctuations of baryon number charge & strangeness in (2+1)-flavor QCD

RBC-Bielefeld, preliminary

vanishing chemical potentials:



$$\chi_2^Q = \frac{1}{VT^3}\langle Q^2\rangle$$

$$\chi_2^B = \frac{1}{VT^3}\langle N_B^2\rangle$$

$$\chi_2^S = \frac{1}{VT^3}\langle N_S^2\rangle$$

rapid approach to SB limit

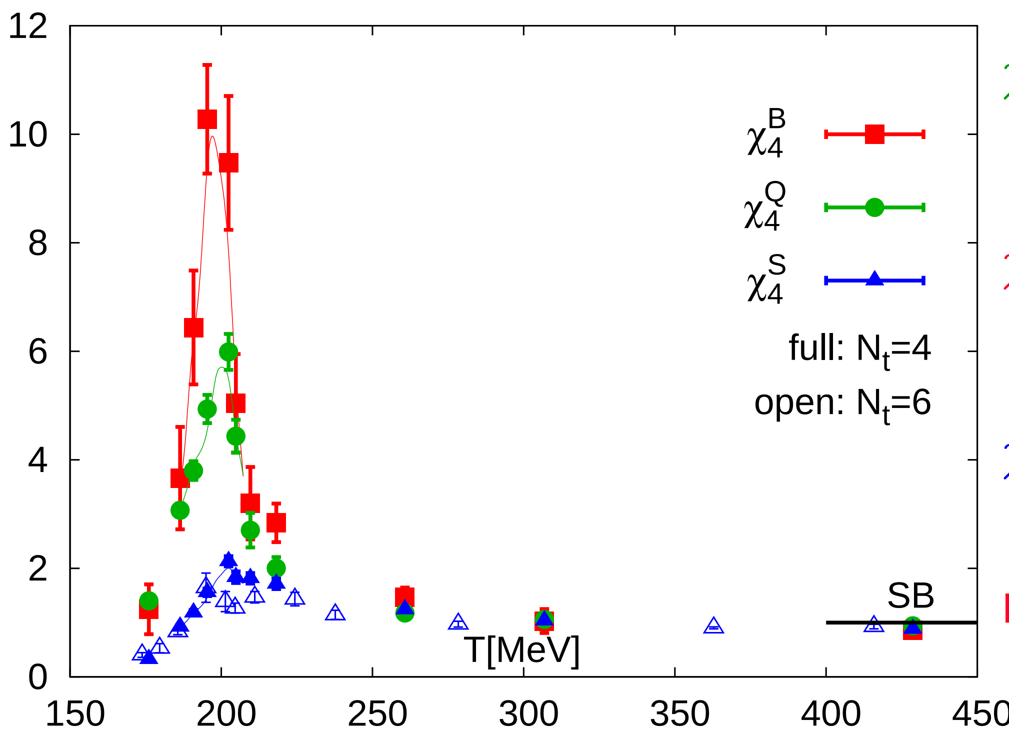
$\Rightarrow$ smooth change of quadratic fluctuations across transition region

chiral limit: $\chi_2^B,\ \chi_2^Q \sim |T - T_c|^{1-\alpha} + \mathbf{regular}$

Quartic fluctuations of baryon number charge & strangeness in (2+1)-flavor QCD

RBC-Bielefeld, preliminary

vanishing chemical potentials:



$$\chi_4^Q = \frac{1}{VT^3}\left(\langle Q^4\rangle - 3\langle Q^2\rangle^2\right)$$

$$\chi_4^B = \frac{1}{VT^3}\left(\langle N_B^4\rangle - 3\langle N_B^2\rangle^2\right)$$

$$\chi_4^S = \frac{1}{VT^3}\left(\langle N_S^4\rangle - 3\langle N_S^2\rangle^2\right)$$

rapid approach to SB limit

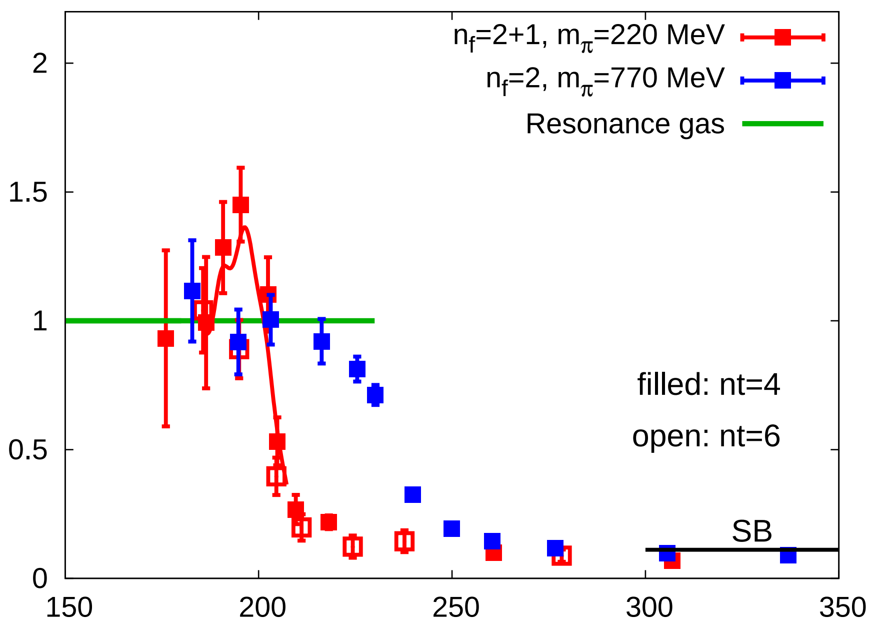
$\Rightarrow$ large light quark number & charge fluctuations across transition region

chiral limit: $\chi_4^B,\ \chi_4^Q \sim |T - T_c|^{-\alpha} + \mathbf{regular}$

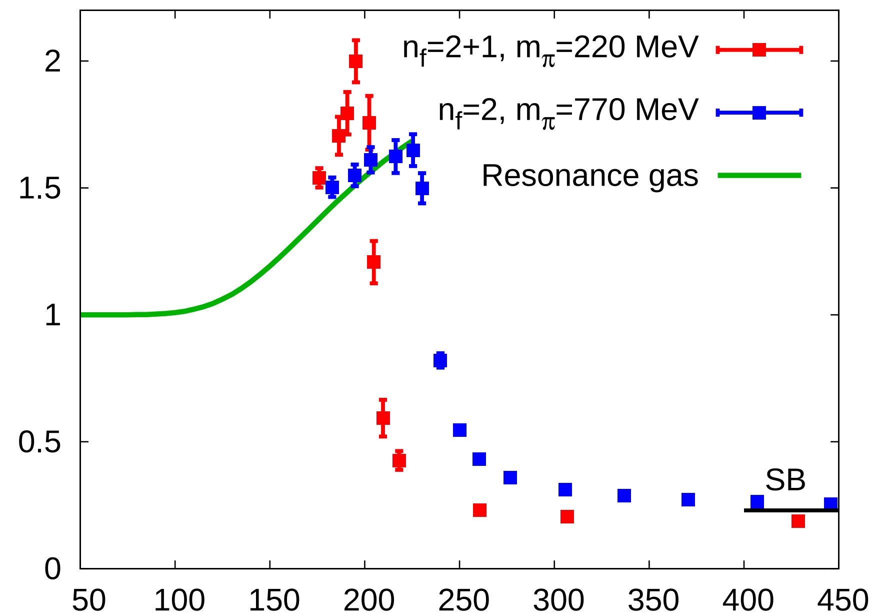
Ratios of quartic and quadratic fluctuations of charges in (2+1)-flavor QCD

$n_f = 2$: S. Ejiri, FK, K.Redlich, PLB633 (2006) 275
$n_f = 2 + 1$: RBC-Bielefeld, preliminary

baryon number fluctuation



charge fluctuation



chiral limit: ratios $\sim |T - T_c|^{-\alpha} + \mathrm{regular}$

$\Rightarrow$ enhancement over resonance gas values? (need to improve $N_\tau = 6$)

$\Rightarrow$ may be observable in event-by-event fluctuations

quark sector quickly ($T \gtrsim 1.5T_c$) behaves perturbative

Quark number in Boltzmann approximation

- baryonic sector of pressure in a hadron resonance gas;

  $m_B \gg T \;\Rightarrow$ Boltzmann approximation: $p_B/T^4 = \sum_{m \le m_{max}} p_m/T^4$

with $$p_m/T^4 = F(T,m,V)\cosh(B\mu_B/T)$$

$$\chi_2^B \equiv \frac{\partial^2 p_m/T^4}{\partial(\mu_B/T)^2} = B^2 F(T,m,V)\cosh(B\mu_B/T)$$

$$\chi_4^B \equiv \frac{\partial^4 p_m/T^4}{\partial(\mu_B/T)^4} = B^4 F(T,m,V)\cosh(B\mu_B/T)$$

ratio of fourth (χ_4^B) and second (χ_2^B) cumulant of quark number fluctuation gives "unit of charge" carried by the particle with mass "m":

$$m \gg T \quad \Rightarrow \quad R_{4,2}^B \equiv \frac{\chi_4^B}{\chi_2^B} = B^2$$

Charge fluctuations in Boltzmann approximation

- hadronic resonance gas: contributions from isosinglet ($G^{(1)}:\ \eta, \ldots$) and isotriplet ($G^{(3)}:\ \pi, \ldots$) mesons as well as isodoublet ($F^{(2)}:\ p, n, \ldots$) and isoquartet ($F^{(4)}:\ \Delta, \ldots$) baryons

$$\begin{aligned}
\frac{p(T,\mu_q,\mu_I)}{T^4} &\simeq G^{(1)}(T) + G^{(3)}(T)\frac{1}{3}\left(2\cosh\left(\frac{2\mu_I}{T}\right)+1\right) \\
&+F^{(2)}(T)\cosh\left(\frac{3\mu_q}{T}\right)\cosh\left(\frac{\mu_I}{T}\right) \\
&+F^{(4)}(T)\frac{1}{2}\cosh\left(\frac{3\mu_q}{T}\right)\left[\cosh\left(\frac{\mu_I}{T}\right)+\cosh\left(\frac{3\mu_I}{T}\right)\right]
\end{aligned}$$

- charge fluctuations at $\mu_q = \mu_I = 0$;
  isospin quartet $F^{(4)}$ contains baryons carrying charge 2

$$R^Q_{4,2} \equiv \frac{\chi^Q_4}{\chi^Q_2} = \frac{4G^{(3)} + 3F^{(2)} + 27F^{(4)}}{4G^{(3)} + 3F^{(2)} + 9F^{(4)}} \rightarrow 1 \text{ for } T \rightarrow 0$$

  contribution of doubly charged baryons increases quartic relative to quadratic fluctuations

Hadronic fluctuations at $\mu_q = 0$

- expect 2^{nd} order transition in 3-d, O(4) symmetry class;

  scaling field: $t = \left|\frac{T-T_c}{T_c}\right| + A\left(\frac{\mu_q}{T_c}\right)^2 \quad , \quad \mu_{crit} = 0$

  singular part: $f_s(T,\mu_u,\mu_d) = b^{-1} f_s(tb^{1/(2-\alpha)}) \sim t^{2-\alpha}$

$$c_2 \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial \mu_q^2} \sim t^{1-\alpha} \quad , \quad c_4 \sim \frac{\partial^4 \ln \mathcal{Z}}{\partial \mu_q^4} \sim t^{-\alpha} \qquad (\mu = 0)$$

- O(4)/O(2): $\alpha < 0$, small $\Rightarrow$

  $c_2 \sim \langle (\delta N_q)^2 \rangle$ dominated by T-dependence of regular part

  $c_4 \sim \langle (\delta N_q)^4 \rangle - 3\langle (\delta N_q)^2 \rangle^2$ develops a cusp

  Y. Hatta, T. Ikeda, PRD67 (2003) 014028

(Hadronic) Fluctuations at $\mu_q > 0$

- expect 2^{nd} order transition in 3-d, O(4) symmetry class;

scaling field: $t = \left|\frac{T - T_c}{T_c}\right| + A\left(\left(\frac{\mu_q}{T_c}\right)^2 - \left(\frac{\mu_{crit}}{T_c}\right)^2\right)$

$$\Uparrow \quad A\left(\frac{\mu_q}{T_c} + \frac{\mu_{crit}}{T_c}\right)\left(\frac{\mu_q}{T_c} - \frac{\mu_{crit}}{T_c}\right)$$

$$\frac{\partial^2 \ln \mathcal{Z}}{\partial \mu_q^2} \sim t^{-\alpha} \quad , \quad \frac{\partial^4 \ln \mathcal{Z}}{\partial \mu_q^4} \sim t^{-(2+\alpha)} \qquad (\mu > 0)$$

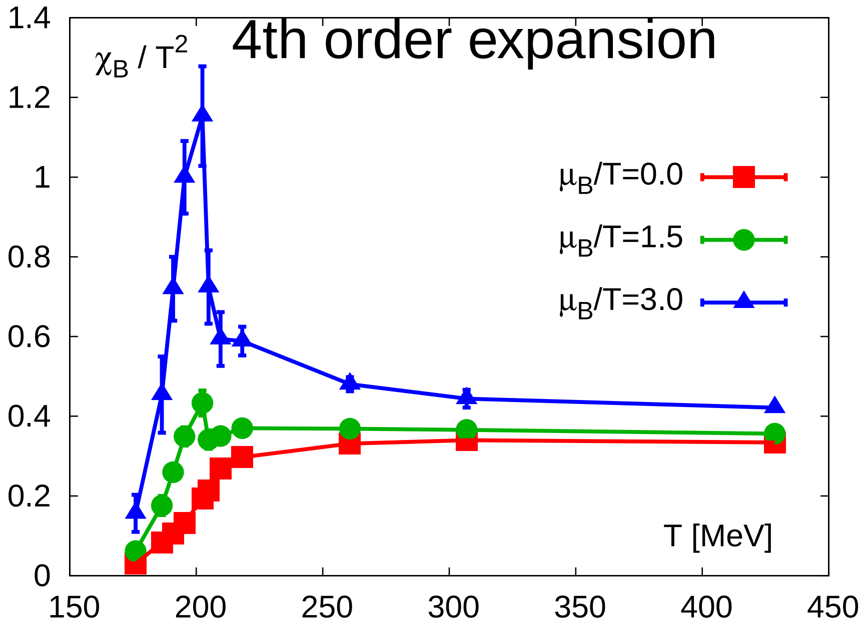
- already second derivative w.r.t. μ_q "looks like a specific heat"

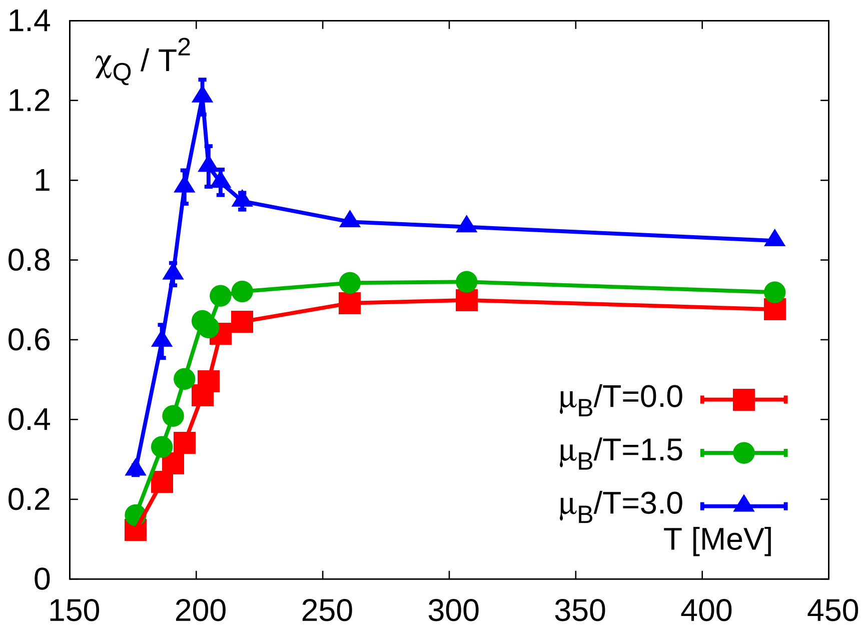
  $\langle(\delta N_q)^2\rangle$ develops a cusp

  $\langle(\delta N_q)^4\rangle$ diverges on the O(4) critical line;

  strength $\sim \left(\frac{\mu_q}{T_c}\right)^4$ ($\sim 10^{-4}$ at RHIC)

Fluctuations of baryon number and charge densities ($\mu \geq 0$)





baryon number density fluctuations:

$$\frac{\chi_B}{T^3} = \left(\frac{\mathrm{d}^2}{\mathrm{d}(\mu_B/T)^2} \frac{p}{T^4} \right)_{T \text{ fixed}}$$

$$= \frac{T}{V} \left(\langle N_B^2 \rangle - \langle N_B \rangle^2 \right)$$

susceptibilities

- to be studied in event-by-event fluctuations
- to be compared to hadron resonance gas phenomenology

seeing "true" singular behavior as signal for a critical point requires large volumes and high order Taylor expansions

$m_\pi = 220$ MeV, (2+1)-flavor QCD

evidence for a critical point??

Isothermal compressibility of the quark gluon plasma

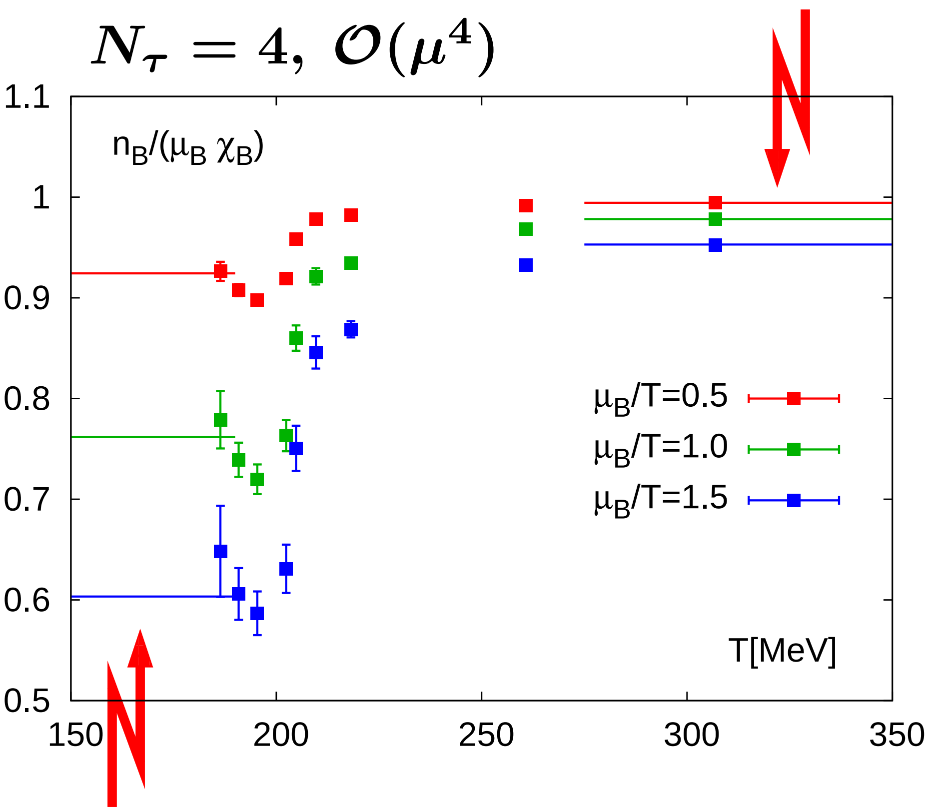
RBC-Bielefeld, preliminary:

$N_\tau = 4,\ 6,\ m_\pi = 220$ MeV

$\mathcal{O}(a^2)$ improved

ideal $q\bar{q}$ gas

$N_\tau = 4,\ \mathcal{O}(\mu^4)$



resonance gas

inverse compressibility:

$$\kappa_T^{-1} = \frac{n_B}{\chi_B} = \left(\frac{\partial p}{\partial n_B}\right)_{T\ \mathrm{fixed}}$$

high-T, massless limit: polynomial in (μ_B/T)

$$\frac{n_B}{\chi_B} = \mu_B + \mathcal{O}\left(\left(\frac{\mu_B}{T}\right)^3\right)$$

large density fluctuations for $\mu_B > 0,\ T < T_c$

"saturated" by fluctuations in a hadron resonance gas

expect: $$\left(\frac{\partial p}{\partial n_B}\right)_T = \frac{n_B}{\chi_B} = 0$$

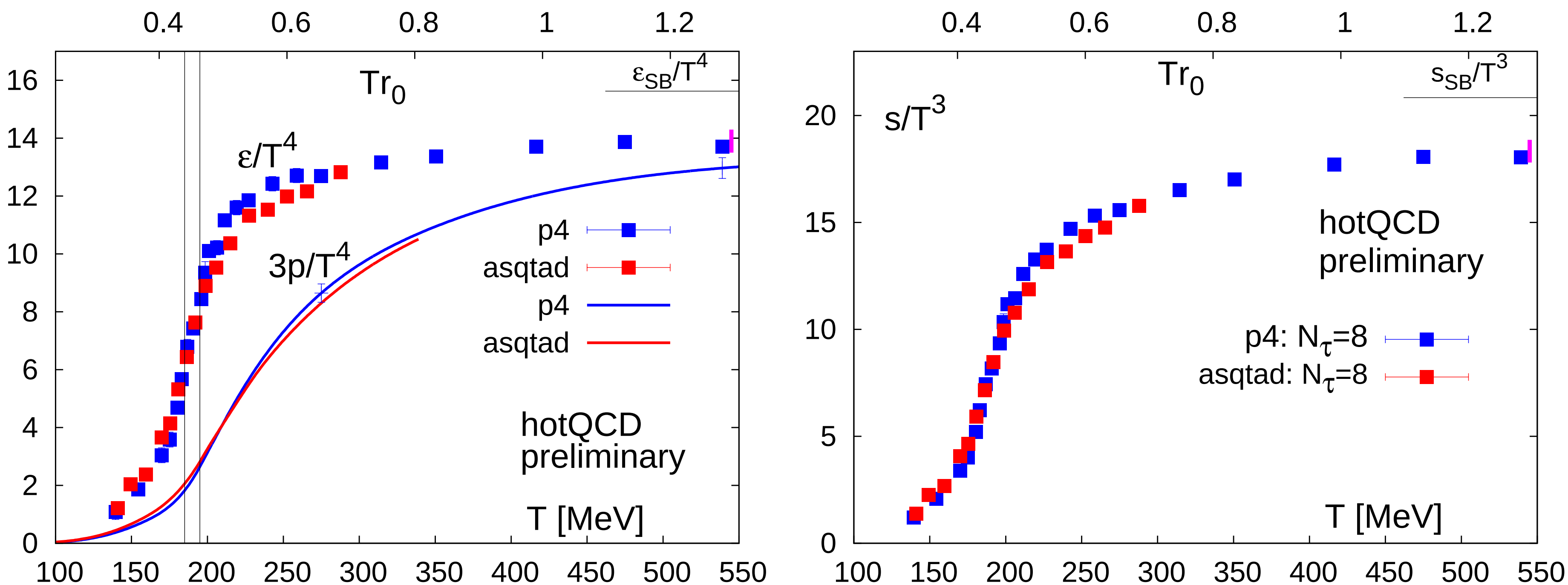
at chiral critical point

Conclusions

- calculations for non-vanishing chemical potential ($\mu_q > 0$) show a rapid transition from a HRG to a QGP; signaled by sudden changes in EoS and susceptibilities
- fluctuations on the crossover line increase with increasing baryon chemical potential in accordance with expectations based on a hadron resonance gas model
  - evidence for additional singular contributions on top of this, which would give direct evidence for the existence of a critical point, is still weak ($\mathcal{O}(\mu^6)$ contribution?)
- Need to reduce statistical uncertainty in higher order expansion coefficients to substantiate evidence for the existence of a critical point - within the Taylor expansion approach this is just a question of computing power
- a large critical value, $\mu_B^{crit}/T > 2$, is favored

Pressure, Energy and Entropy

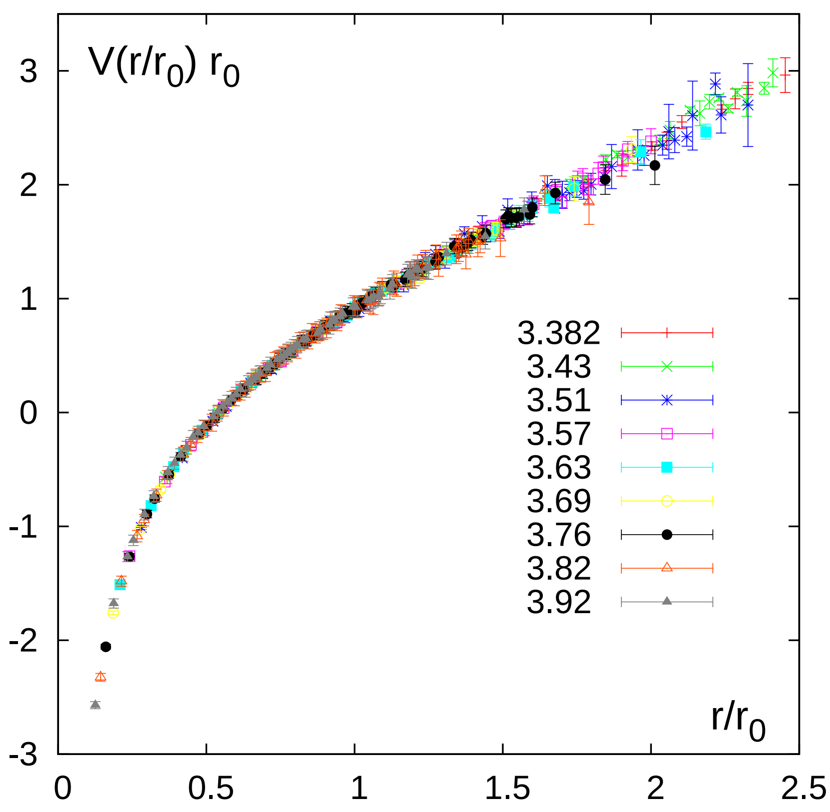
- p/T^4 from integration over $(\epsilon - 3p)/T^5$;
  using piecewise quadratic fit with $T_0 = 100$ MeV with $p(T_0) = 0$;
- systematic error on $3p/T^4 \simeq 0.33$



$T = 0$ scale setting using the heavy quark potential

use r_0 or string tension to set the scale for $T = 1/N_\tau a(\beta)$

$$V(r) = -\frac{\alpha}{r} + \sigma r \quad , \quad r^2 \frac{\mathrm{d}V(r)}{\mathrm{d}r}\Big|_{r=r_0} = 1.65$$



no significant cut-off dependence when cut-off varies by a factor 5

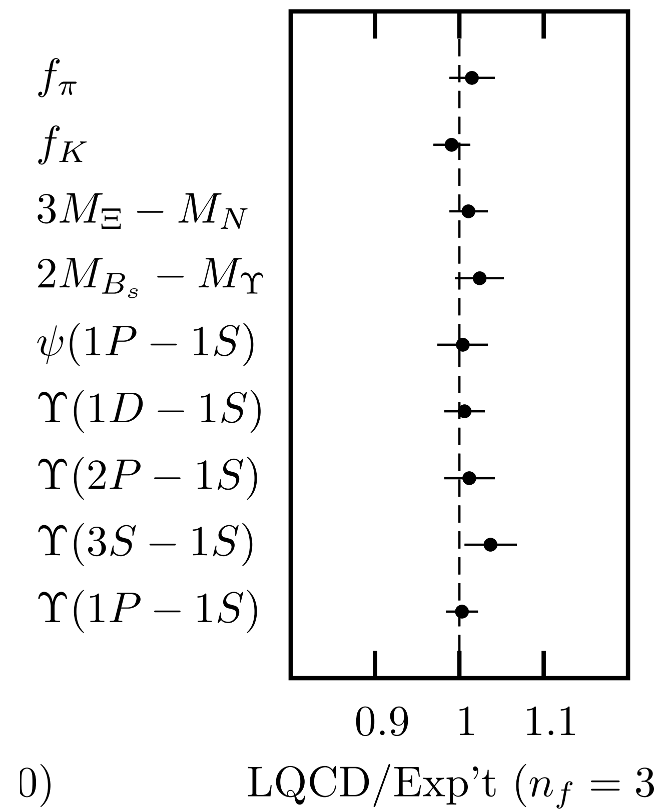
i.e. from the transition region on $N_\tau = 4$ lattices ($a \simeq 0.25$ fm) to that on $N_\tau = 20$ lattices ($a \simeq 0.05$ fm) !!

scales extracted from 'gold plated observables'

- high precision studies of several experimentally well known observables in lattice calculations with staggered (asqtad) fermions led to convincing agreement $\Rightarrow$ gold plated observables
- simultaneous determination of r_0/a in these calculations determines the scale r_0 in MeV

knowing any of these experimentally accessible quantities accurately from a lattice calculation is equivalent to knowing r_0, which is a fundamental parameter of QCD

C.T.H. Davies et al., PRL 92 (2004) 022001
A. Gray et al., PRD72 (2005) 094507



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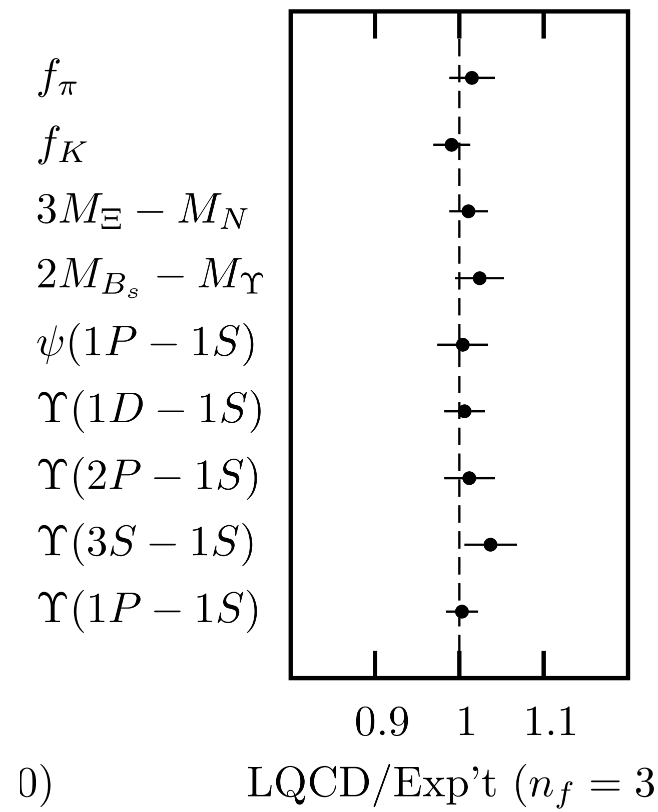
- simultaneous determination of r_0/a in these calculations determines the scale r_0 in MeV

we use $r_0 = 0.469(7)$ fm determined from quarkonium spectroscopy
A. Gray et al, Phys. Rev. D72 (2005) 094507



C.T.H. Davies et al., PRL 92 (2004) 022001
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