Scaling and Finite-Size Scaling analysis of critical behavior in lattice QCD

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J. Braun and B. Klein, Phys. Rev. D **77** (2008) 096008;
PoS **LAT2007**:198 (2007) and in preparation (2008).

Technische Universität München

Overview

Phase transitions in QCD

QCD phase transitions

- de-confinement phase transition
- chiral phase transition

Phase transitions in QCD

QCD phase transitions

- de-confinement phase transition
- chiral phase transition

Lattice Gauge Theory

- fully non-perturbative method
- finite simulation volume
- explicit symmetry breaking through quark masses
- phase transition order? $\longrightarrow$ (finite-size) scaling analysis

Phase transitions in Finite Volume

- no actual phase transition in finite volume
- thermodynamic potential completely regular
- order parameter vanishes strictly ...
- ...unless explicit symmetry breaking is present
- critical behavior difficult to identify

finite-size scaling is an essential tool for the analysis of critical behavior in finite-volume simulations

- $N_f = 2$: second order for $m_q = 0$, crossover for $m_q \neq 0$, O(4)

R. D. Pisarski and F. Wilczek, Phys. Rev. D **29** (1984) 338.

- first order phase transition? (confinement dominates?) (staggered fermions)

M. D'Elia, A. Di Giacomo and C. Pica, Phys. Rev. D **72** (2005) 114510 [arXiv:hep-lat/0503030];

G. Cossu, M. D'Elia, A. Di Giacomo, and C. Pica (2007), arXiv:0706.4470 [hep-lat].

- consistent with O(2) for χQCD (staggered) (but no scaling)

J. B. Kogut and D. K. Sinclair, Phys. Rev. D **73** (2006) 074512 [arXiv:hep-lat/0603021].

- decide by analyzing scaling behavior

A scaling analysis requires *critical exponents* and *scaling functions*

- Scaling functions obtained mostly from O(N) lattice simulations and perturbative RG

  D. Toussaint, Phys. Rev. D55 (1997) 362.

  J. Engels, S. Holtmann, T. Mendes, and T. Schulze, Phys. Lett. **B514** (2001) 299.

  E. Brézin, D. J. Wallace, and K. Wilson, Phys. Rev. B7, 232 (1973).

  F. Parisen Toldin, A. Pelissetto and E. Vicari, JHEP **0307** (2003) 029.

- O(N) critical exponents from FRG calculations

  N. Tetradis and C. Wetterich, Nucl. Phys. **B422** (1994) 541.

  O. Bohr, B.J. Schaefer, and J. Wambach, Int. J. Mod. Phys. **A16** (2001) 3823.

  D. F. Litim and J. M. Pawlowski, Phys. Lett. B **516** (2001) 197.

- few results with explicit symmetry breaking
- no results on finite-size scaling

Functional RG for the O(N)-model

Effective action at scale k ($\Lambda \geq k \geq 0$) in LPA ($\eta = 0$)

$$\Gamma_k[\phi] = \int d^d x \, \frac{1}{2}(\partial_\mu \phi)^2 +$$

$$+a_1(k)(\phi^2 - \sigma_0^2(k)) + a_2(k)(\phi^2 - \sigma_0^2(k))^2 + \ldots - H(\sigma - \sigma_0(k))$$

$$\phi = (\sigma, \vec{\pi}) \qquad \phi^2 = \sigma^2 + \vec{\pi}^2 \quad O(N)\text{-symmetric}$$

scale-dependent couplings

$\sigma_0(k), a_n(k), \quad n = 1, \ldots, n_{max} \qquad 2a_1(k)\sigma_0(k) = H \equiv const.$

Inputs:

- choice of initial scale Λ sets all scales
- couplings $\sigma_0(\Lambda)$ and $a_2(\Lambda)$ at initial scale Λ
- $d = 3$: no field-theoretical temperature
- $\sigma_0(\Lambda) - \sigma_0^{\mathrm{crit}}(\Lambda) \sim T - T_c$:
  $\sigma_0(\Lambda) > \sigma_0^{\mathrm{crit}}(\Lambda) \Rightarrow$ broken phase
  $\sigma_0(\Lambda) < \sigma_0^{\mathrm{crit}}(\Lambda) \Rightarrow$ symmetric phase

application to finite volume:

$$\int \frac{d^d p}{(2\pi)^d} \; f(p^2) \longrightarrow \frac{1}{L^d} \sum_{n_1,\ldots,n_d} f\left(\left(\frac{2\pi}{L}\right)^2 (n_1^2 + \ldots + n_d^2)\right)$$

- local expansion around the minimum
- lowest order in local potential approximation: $\eta = 0$
- potential with explicit symmetry breaking
- ERG with optimized cutoff D. F. Litim, Phys. Lett. B486 (2000) 92.
  $\Leftrightarrow$ proper-time RG (infinite volume) S. B. Liao, Phys. Rev. D53 (1996) 2020.

  D. F. Litim and J. M. Pawlowski, Phys. Lett. B 516 (2001) 197.

Scaling for the singular free energy

there is no length scale at a critical point: $\xi \to \infty$
▷ scale invariance of free energy density at critical point
▷ use behavior close to critical point

$$f_s(t, h) = \ell^{-d} f_s(\ell^{y_t} t, \ell^{y_h} h)$$

$$t = (T - T_c)/T_0, \quad h = H/H_0, \quad y_t = \frac{1}{\nu}, \quad y_h = \frac{\nu}{\beta\delta}$$

idea: keep one of the arguments fixed
▷ becomes function of a single scaling variable

Scaling function in Fisher parametrization

$$M \sim \frac{\partial}{\partial H} f(t, h)$$

$$t = (T - T_c)/T_0, \qquad h = H/H_0, \qquad z = \frac{t}{h^{1/(\beta\delta)}}, \qquad \xi(t) \sim t^{-\nu}$$

Scaling function for the order parameter M

$$\left.\begin{array}{l} M(t, h=0) = (-t)^{\beta} \\ M(t=0, h) = \quad h^{1/\delta} \end{array}\right\} \to M(t,h) = h^{1/\delta} f(z), \;\; f(z) \xrightarrow[z\to-\infty]{} (-z)^{\beta}$$

critical exponent β enters into asymptotic behavior

Scaling function in Fisher parameterization

Scaling function for the susceptibility χ

$$\chi = \frac{\partial M}{\partial H} \Rightarrow \chi = \frac{1}{H_0} h^{1/\delta - 1} \frac{1}{\delta}\left[f(z) - \frac{z}{\beta} f'(z) \right] = \frac{1}{H_0} h^{1/\delta - 1} f_\chi(z)$$

▷ consequence of scaling:

$f_\chi(z)$ completely specified by $f(z)$

Critical exponents for the O(4) model in d=3

	ν	β	η	δ
lattice[1]	0.7423	0.380	0.024*	4.86
RG[2]	0.8043	0.4022*	–	5.00*
RG	0.8053(60)	0.4030(30)	–	4.973(30)

[1] J. Engels, S. Holtmann, T. Mendes and T. Schulze, Phys. Lett. B **514** (2001) 299.

[2] D. F. Litim and J. M. Pawlowski, Phys. Lett. B **516** (2001) 197.

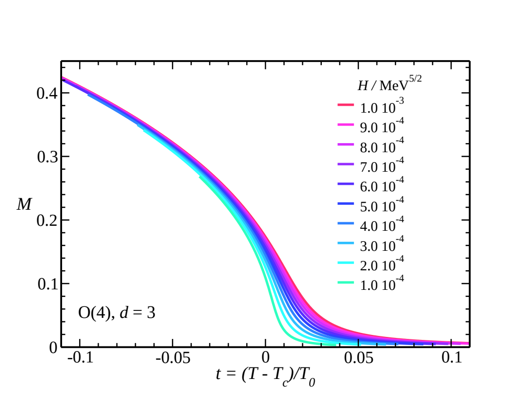
If we already know the critical exponents, why do we want the scaling functions as well?

$$\beta\delta/\nu = (\gamma+\beta)/\nu = (2-\eta) + \frac{1}{2}(d-2+\eta) \approx \frac{5}{2}$$

$$\beta/\nu = \frac{1}{2}(d-2+\eta) \approx \frac{1}{2}$$

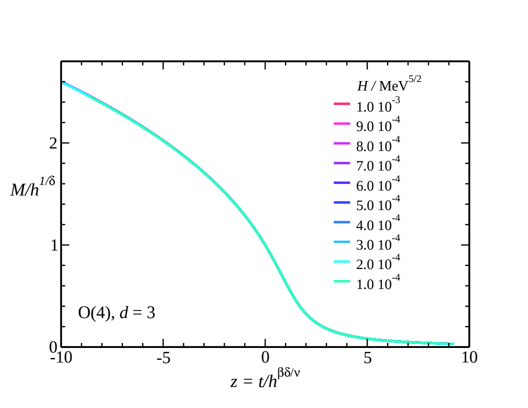
Order parameter M as a function of t

for small values of H

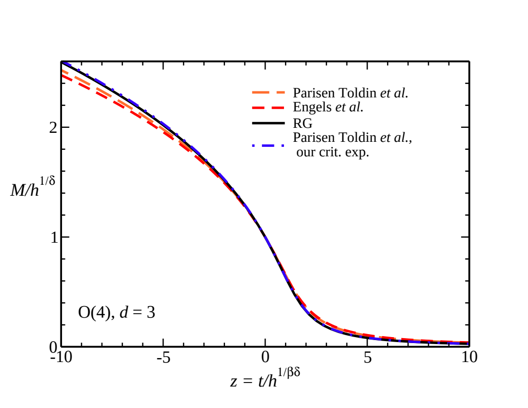


Rescaled order parameter $M/h^{1/\delta}$ as a function of z

for small values of H



Comparison to lattice spin-model O(4) result

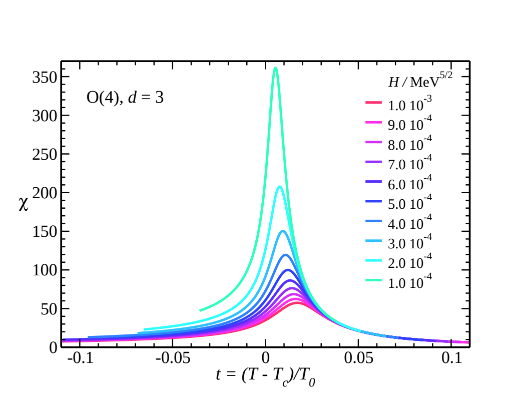


J. Engels, S. Holtmann, T. Mendes, and T. Schulze, Phys. Lett. **B514** (2001) 299

F. Parisen Toldin, A. Pelissetto and E. Vicari, JHEP **0307** (2003) 029.

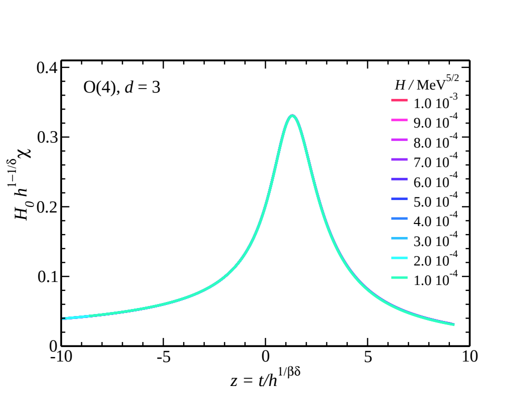
Susceptibility χ as a function of t

for small values of H (note trajectory $t_p/h^{1/(\beta\delta)} = z_p$)



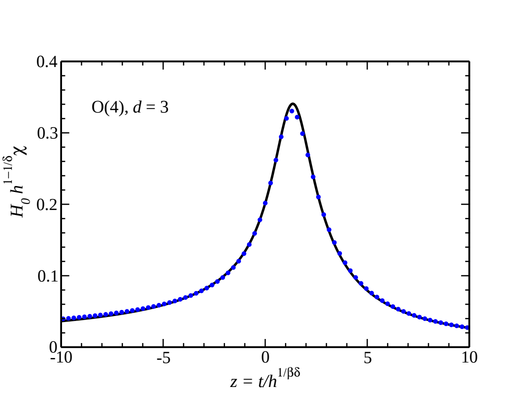
Rescaled Susceptibility $H_0 h^{1-1/\delta}\chi$ as a function of z

for small values of H



Test of scaling function

Is it true that $f_\chi(z) = \frac{1}{\delta}\left[f(z) - \frac{z}{\beta}f'(z)\right]$? Yes!



Finite-Size Scaling

- Universal behavior requires divergence of the correlation length ξ
- Finite volume size L cuts off the critical fluctuations
- Universal scaling behavior is therefore affected if correlation length $\xi \sim L$, depends on ratio ξ/L

Finite-Size Scaling hypothesis (Fisher): The ratio of thermodynamic quantities $(M, \chi, \ldots)$
in the finite-size system and the infinite-size system is a function of *only* the ratio ξ/L:

$$\frac{M_L(t)}{M_\infty(t)} = \mathcal{F}\left(\frac{L}{\xi(t)}\right), \quad \xi(t) \sim t^{-\nu}, \quad M(t,h) = h^{1/\delta} f(z)$$

Finite-Size Scaling Functions

Idea for obtaining the universal Finite-Size Scaling functions:

- keep $L/\xi = \text{const.} \quad \rightarrow \quad \text{vary} \quad t \sim L^{1/\nu}$
- keep $z = t/h^{1/(\beta\delta)} = \text{const.} \quad \rightarrow \quad \text{vary} \quad h \sim L^{-\beta\delta/\nu}$

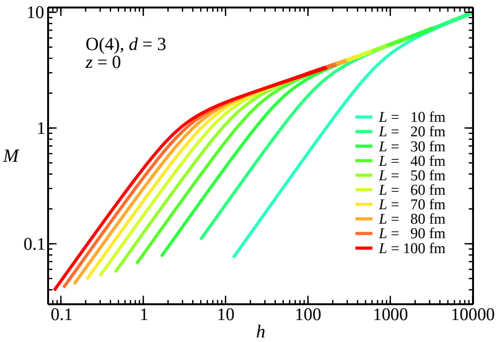
$$M(t,h) = h^{1/\delta} f(z) \quad \rightarrow \quad L^{-\beta/\nu}(hL^{\beta\delta/\nu})^{1/\delta} f(z)$$

Finite-Size Scaling Functions depend only on $hL^{\beta\delta/\nu}$ (for any given value of z):

$$\begin{aligned} L^{\beta/\nu} M &= Q_M(z, hL^{\beta\delta/\nu}) \\ L^{\gamma/\nu}\chi &= Q_\chi(z, hL^{\beta\delta/\nu}) \end{aligned}$$

Finite-Size Scaling

Order parameter $M(h)$ vs. h for $L = 10 - 100$ fm

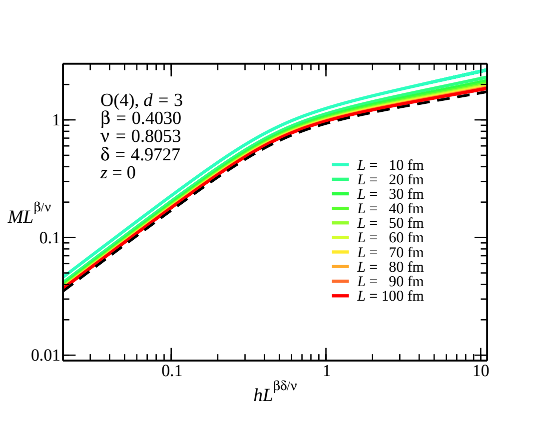


at the critical temperature ($z = 0$).

- ξ small for large h
- deviations for $\xi \sim L$
- asymptotic behavior given by $1/\delta$

Finite-Size Scaling

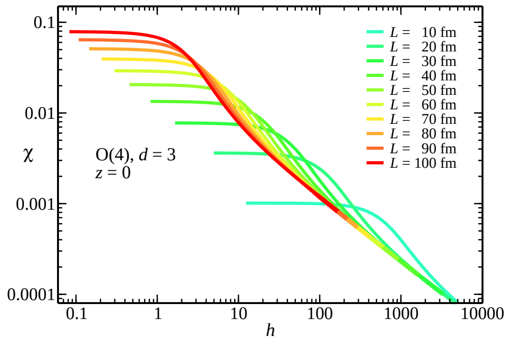
Finite-size scaled order parameter $ML^{\beta/\nu}$ vs. $hL^{\beta\delta/\nu}$



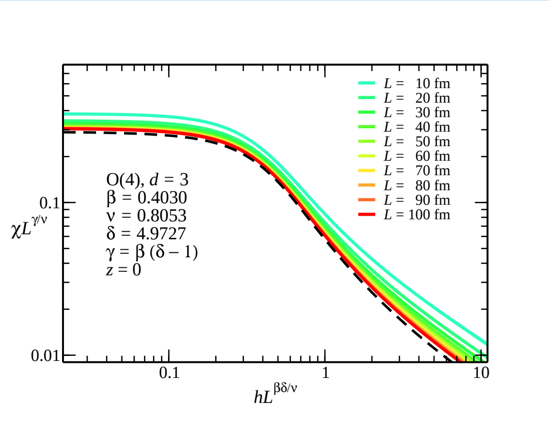
for $L = 10 - 100$ fm.

- scaling deviations for large fields h
- controlled by sub-leading operator
- consistent with RG prediction for ω
- extrapolate to obtain scaling function

Susceptibility $\chi(h)$ vs. h for $L = 10 - 100$ fm

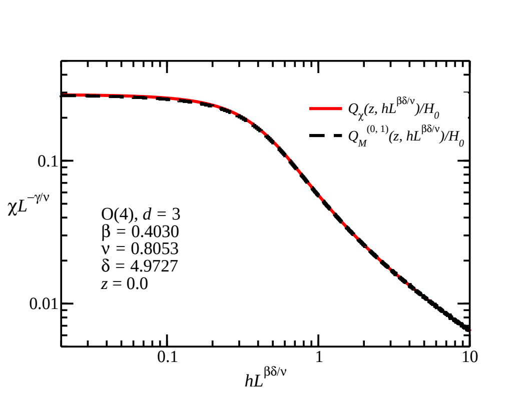


Finite-size scaled susceptibility $\chi L^{\gamma/\nu}$ vs. $hL^{\beta\delta/\nu}$



Test of the scaling function

Is it true that $Q_\chi(z, hL^{\beta\delta/\nu}) = \frac{\partial}{\partial(hL^{\beta\delta/\nu})} Q_M(z, hL^{\beta\delta/\nu})$? Yes!



Comparison to lattice QCD results

exploit *universality* near a critical point to analyze behavior

Issues to address

- Comparison of universal scaling functions
- only IR quantities can be compared, different UV physics
- determine non-universal normalization factors *or*
- determine dimensionless ratios
- locate finite-size scaling regions
- estimate scaling corrections

Scale normalizations

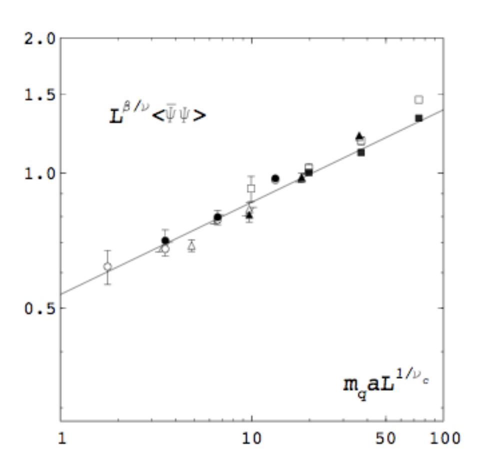
- for every coupling $(t, h, \ldots)$ there is a normalization scale $(T_0, H_0, \ldots)$
- $\bar{h} = hL^{\beta\delta/\nu}$ is not dimensionless $\Rightarrow$ not universal
- need universal quantity: Length scale normalization required

best candidate: dimensionless ratio/product of IR quantities

$$\frac{L}{\xi(t, h, L)} = M_\sigma(t, h, L)L \geq M_\pi(t, h, L)L$$

$$\xi(t) = \frac{1}{C_0} t^{-\nu}, \quad M_\sigma(t) = C_0\, t^{\nu} \Rightarrow \frac{L}{\xi(t, L \to \infty)} = \frac{C_0 L}{t^{-\nu}} = \ell\, t^{\nu}$$

Asymptotic finite-size scaling on the lattice

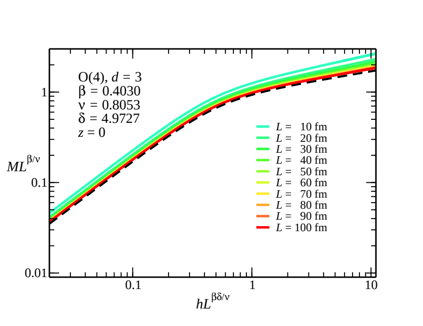


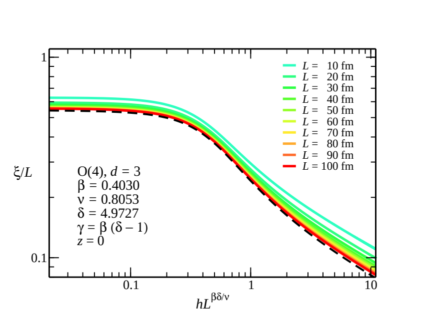
J. Engels, S. Holtmann, T. Mendes, and T. Schulze, Phys. Lett. **B514** (2001) 299.

- asymptotic behavior for $hL^{\beta\delta/\nu}$ large
- no large finite-size scaling effects in asymptotic region
- lattice-N_s is dimensionless, but $N_s = L/a(O(N)) \neq N_s = L/a(QCD)$: need to fix ξ/L

Finite-Size Scaling regions

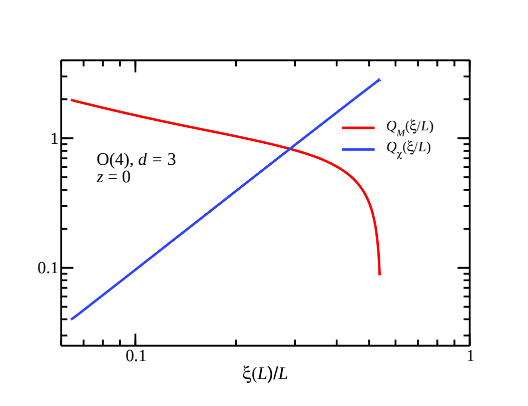
- pion mass provides bound for ξ/L: $m_\pi \leq \frac{1}{\xi} \Leftrightarrow m_\pi L \leq \frac{L}{\xi}$
- $m_\pi L$ must not be too large to see sizable effects





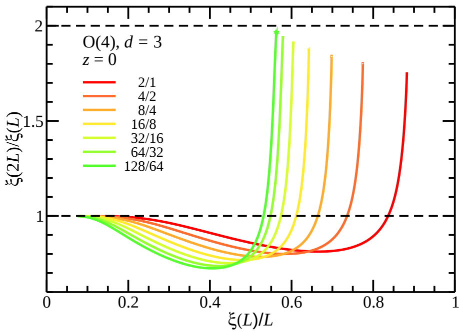
Observables as a function of ξ/L

scaling functions plotted against the dimensionless ratio $\xi(L)/L$



Ratios of results from systems with fixed size ratio

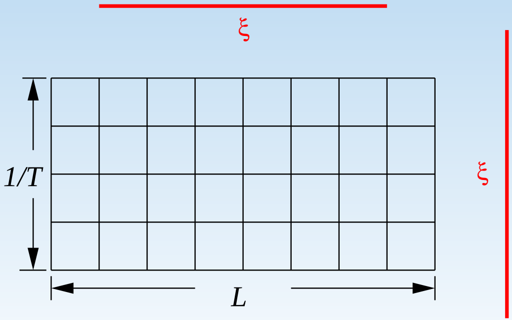
$\xi(2L)/\xi(L)$ as a function of $\xi(L)/L$



- no normalization problems
- but large corrections
- correlation length bounded $\xi(L) \leq k_0 L$
- ratio approaches volume ratio for $L \to \infty$

Dimensional reduction

When can we expect scaling behavior in a $d = 3$ class?



- for $d = 3$ behavior need $\xi \gg 1/T$
- for infinite-volume scaling behavior need $\xi/L \ll 1$
- for finite-size scaling need $\xi/L \approx 1$
- $m_\pi L \leq L/\xi$ is bound

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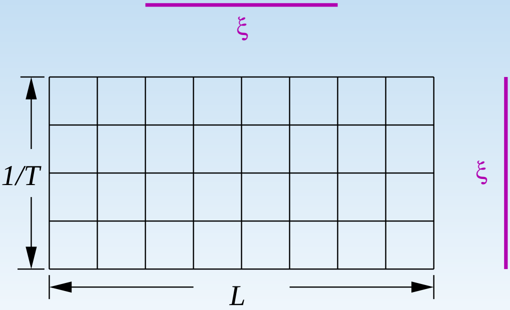

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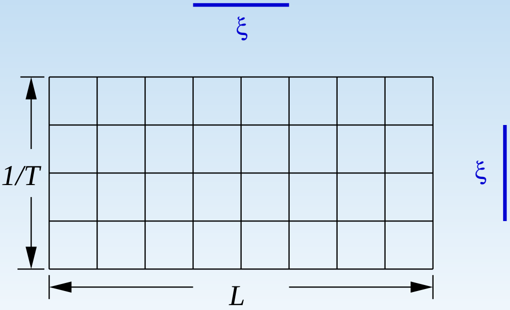

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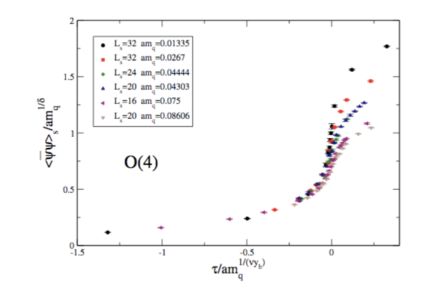
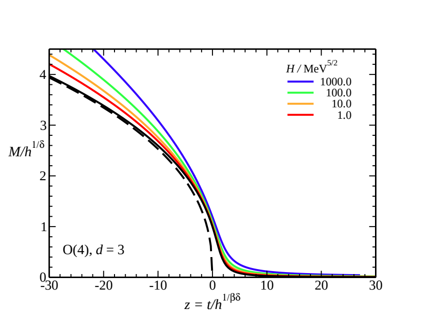
Scaling of the order parameter on the lattice

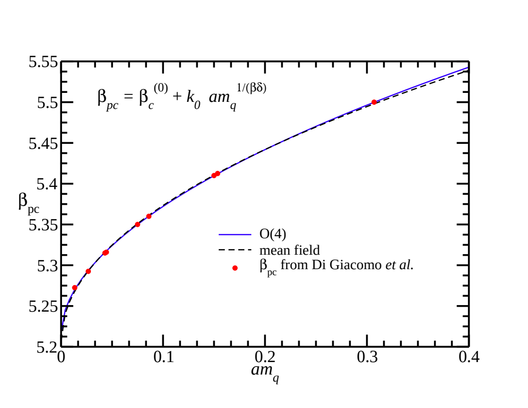
- rescaled order parameter (subtracted) from DiGiacomo et al.
- scaling corrections largest in phase with broken symmetry





M. D'Elia, A. Di Giacomo and C. Pica, Phys. Rev. D **72** (2005) 114510 [arXiv:hep-lat/0503030]

pseudocritical coupling on the lattice



what is the relevant T?

$T = \frac{1}{N_t a(\beta, am_q)}$

$\Rightarrow \frac{T_c - T}{T_c} = 1 - \frac{a(\beta_c^{(0)}, 0)}{a(\beta, am_q)}$

what is the relevant h?

$h = am_q N_t$

($h = \beta H$ in spin systems)

submission: use β and am_q

M. D'Elia, A. Di Giacomo and C. Pica, Phys. Rev. D **72** (2005) 114510 [arXiv:hep-lat/0503030]

Conclusions

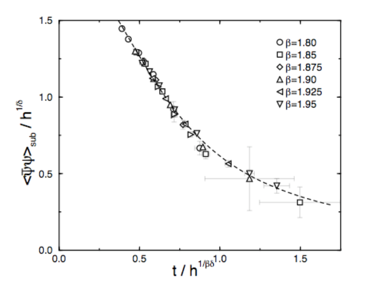
- Scaling and Finite-Size Scaling functions for O(N) class
- relevant FSS region for lattice QCD likely at *smaller* lattices or *smaller* m_π
- large corrections to scaling for large symmetry breaking
- strategy for comparison

Outlook

- comparison to lattice QCD results:
  are calculations in the "interesting" region?
- comparison to O(2) scaling functions

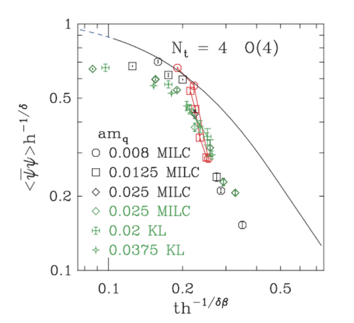
Scaling on the lattice: Staggered vs. Wilson

there is a discrepancy in the $N_f = 2$ scaling behavior between staggered and Wilson fermions:



CP-PACS, A. Ali Khan *et al.*, Phys. Rev. D **63** (2001) 034502

[hep-lat/0008011]



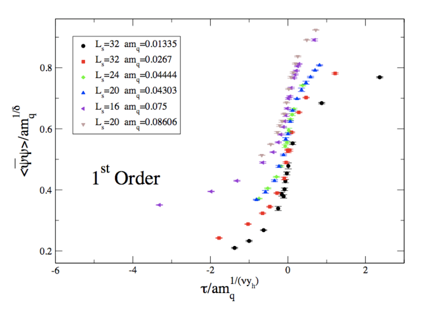
MILC, C. W. Bernard *et al.*, Phys. Rev. D **61** (2000) 054503

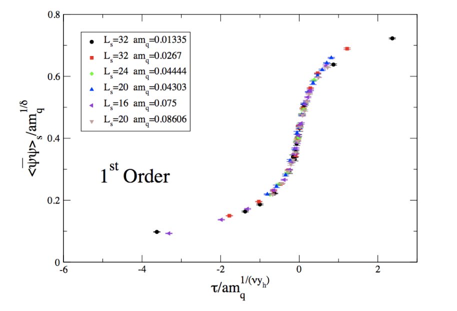
[hep-lat/9908008]

still unresolved!

Scaling of the order parameter on the lattice

- rescaled order parameter (subtracted) from DiGiacomo et al.





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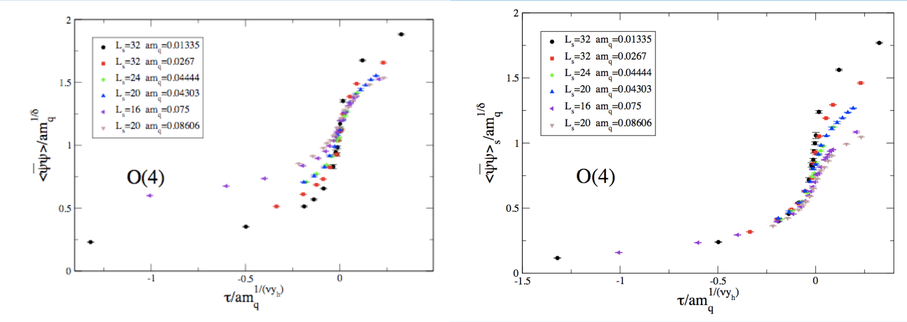

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