The critical endpoint at imaginary μ_B

INT-08-02b The QCD Critical Point

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Based on
M. D'Elia , F. Di Renzo, M. P. Lombardo, Phys. Rev. D 2007 ; QM2008
M.D'E, F.D.R, MpL, A. Vuorinen, work in progress

PLAN

- Approach to the free gas : perturbative and nonperturbative
- The phase diagram in the T, complex μ plane
- The phase of the determinant in the QGP
  or, When the sign problem it is not a problem –
- The endpoint at imaginary μ
- Taylor expansion and radius of convergence
- Quasiparticle models and critical behaviour

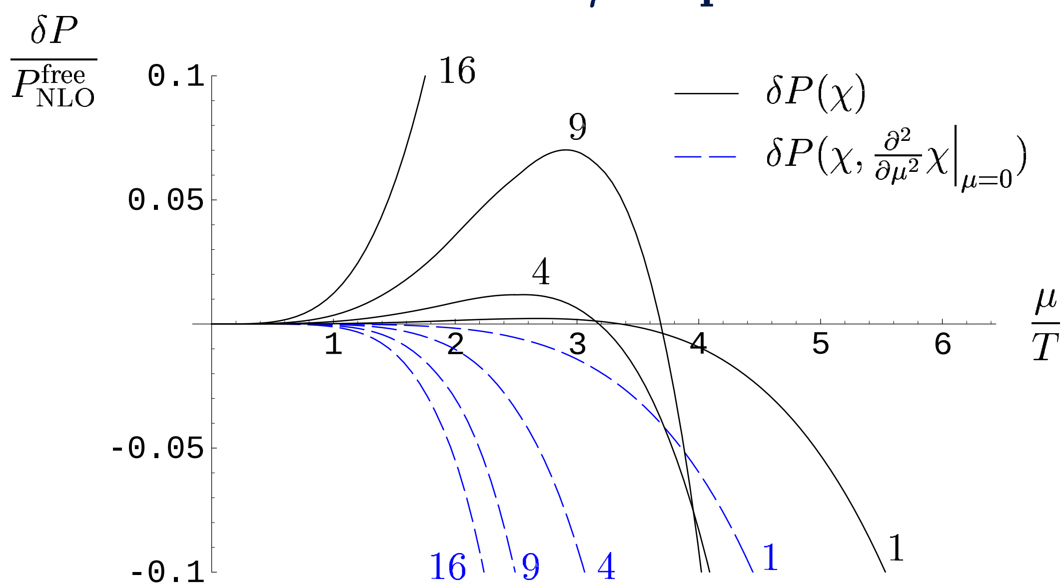
1 Approach to the Stefan-Boltzmann limit

- P(T, μ) approaches *slowly* the free gas
- However, the subtracted pressure
  $\Delta P = P(T, \mu) - P(T, 0)$
  has, apparently, a much faster convergence .
- Monitor: RATIO OF n(μ) TO THE FREE RESULT $n(\mu)_{SB}$

$$R_F(\mu) = n(\mu)/n(\mu)_{SB}$$

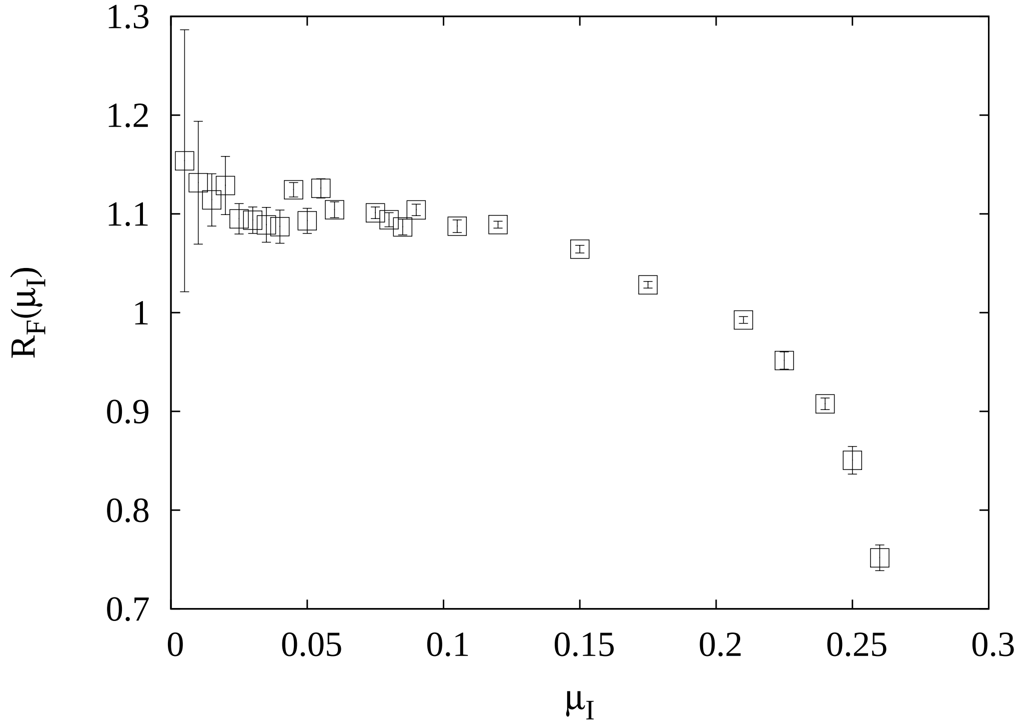
- Free field: $R_F(\mu_I) = 1$

Deviation from a 'trivial' μ dependence at larger μ



A. Ipp and A. Rebhan, 2001

$T = 1.1T_c$



Similar results from

- Taylor expansion:
  see e.g. F. Karsch talk at Extreme QCD , many references
- Hard Loop Thermal Perturbation Theory
  see e.g. J.P. Blaizot, E. Iancu,A. Rebhan Phys.Lett. B 2001; A. Ipp, A. Rebhan, JHEP 2001
- Canonical Ensemble
  S. Kratochvila and P. de Forcrand S. Kratochvila , Phys. Rev. D 2006

Apparent paradhox :

Finite density helps reaching the free field regime

But also

The larger the density the more complicate things become!

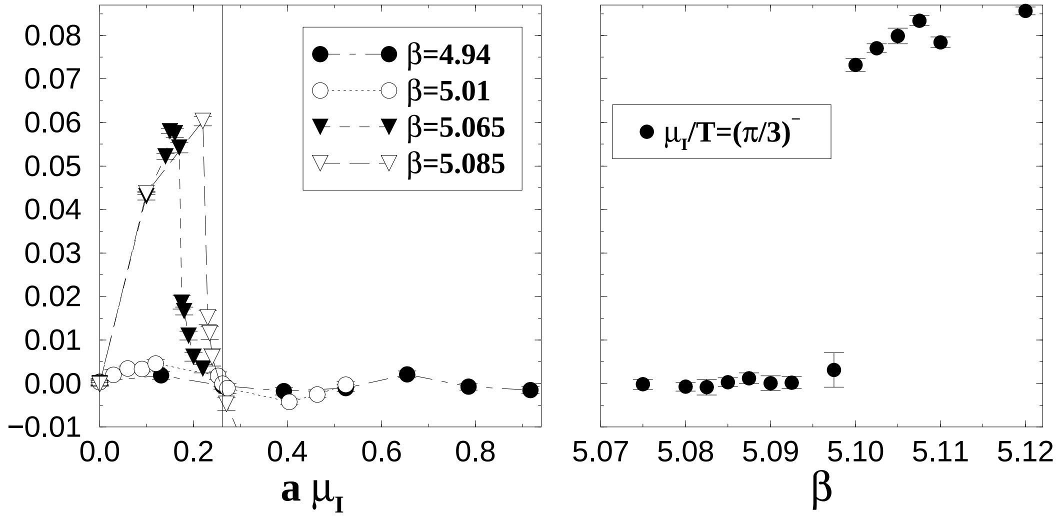
2 The phase diagram in the imaginary μ – temperature space

$$Z(\mu_I/T) \equiv Z(V,T,i\mu_I/T) = \mathrm{Tr}\left(e^{i\mu_I N/T} e^{-\frac{H_{\mathrm{QCD}}}{T}}\right)$$

- N is a number operator: $Z(\mu_I/T)$ periodic in μ_I with period $2T\pi$; moreover a period $2T\pi/3$ is expected in the confined phase, where only physical states with N multiple of 3 are present.
- Observation (Roberge and Weiss) : $Z(\mu_I)$ is always periodic $2T\pi/3$, for any physical temperature!
- Low T : smooth periodicity
- High T : non-analytic behaviour with discontinuities at

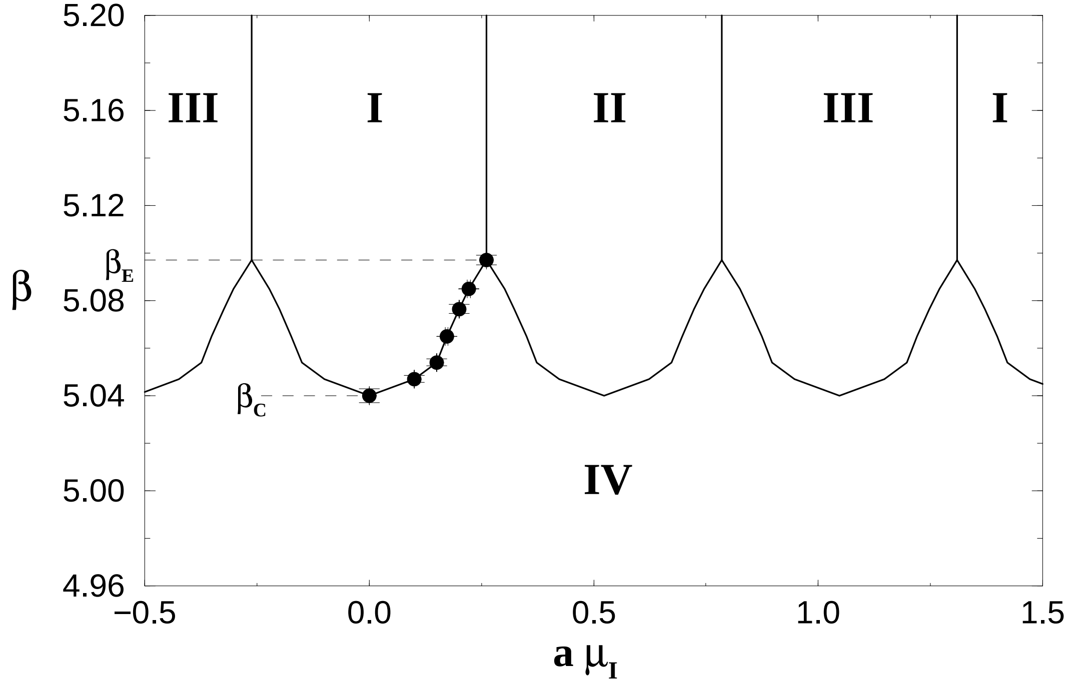
$$\theta = 2\pi/3(k+1/2)$$

  corresponding to phase transitions from one Z_3 sector to the other.
- $P(\vec{x})e^{i\mu_I/T}$, instead of $P(\vec{x})$: μ_I/T fixes the preferred vacuum.

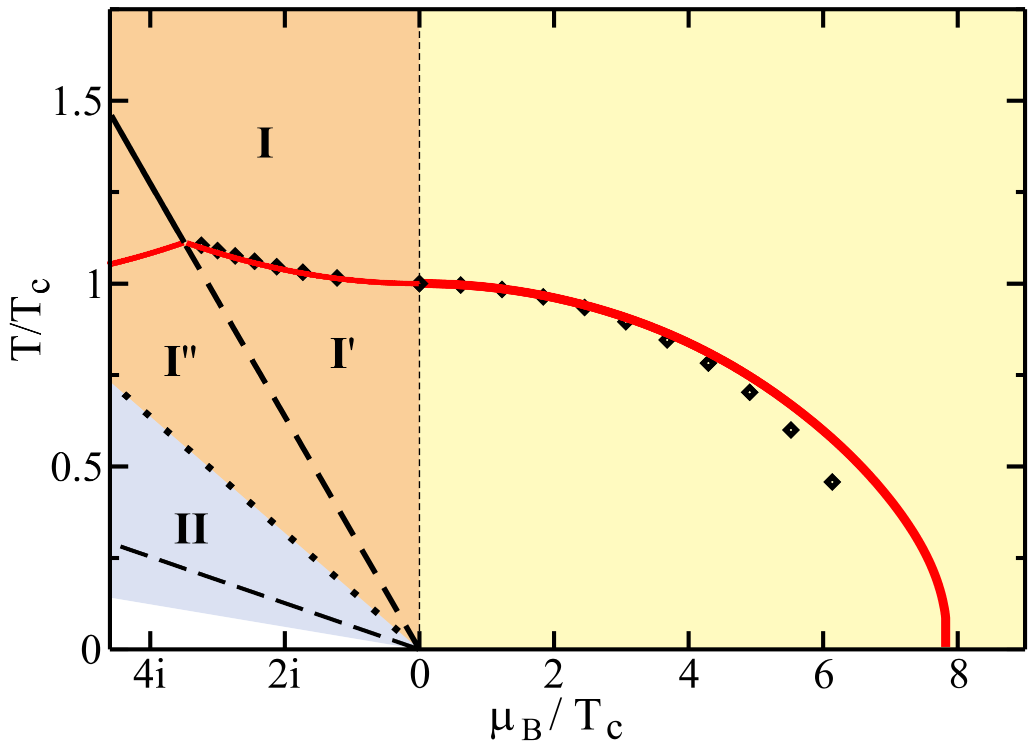


Imaginary part of the barion density as a function of μ_I for different values of β (left–hand side), and as a function of β at $\mu_I/T = \frac{\pi}{3}^-$ (right-hand side).

M. D'Elia, MpL, 2001



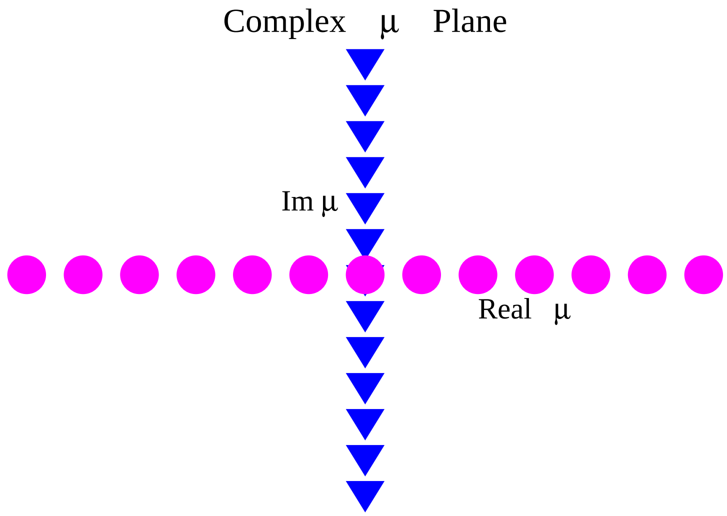
Sketch of the phase diagram in the μ_I–β plane.



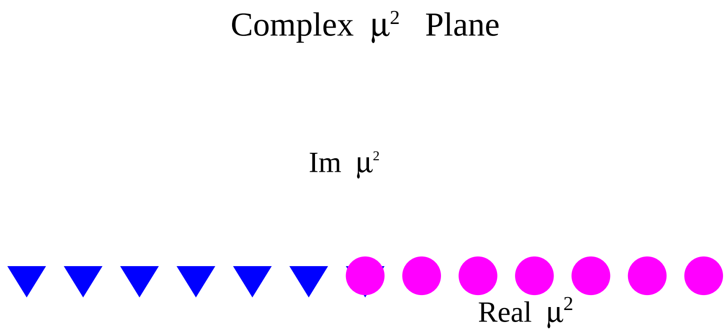
Kämpfer, Bluhm QM08

4 flavor Data from M. D'Elia, MpL 2004
Analytic continuation can be extended at lower T via Pade' (MpL 2005) or phenomenological models (Kampfer Bluhm 2008)

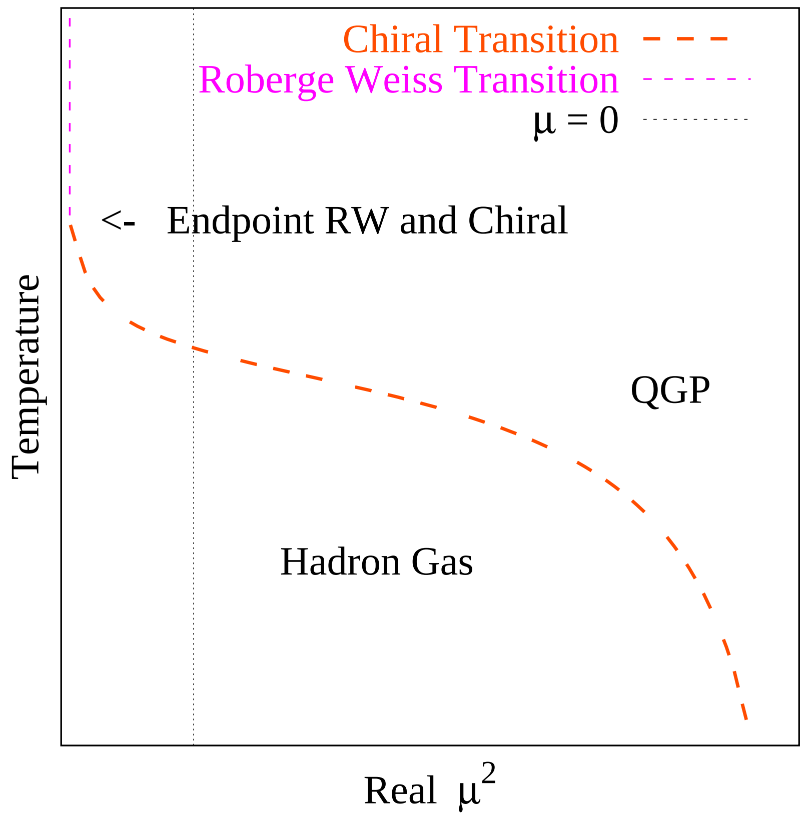
Because of the QCD symmetries, the complex μ_B plane



can be mapped onto the complex μ_B^2 plane



The Phase Diagram in the T, μ_B^2 Plane



3 The phase of the determinant at high T

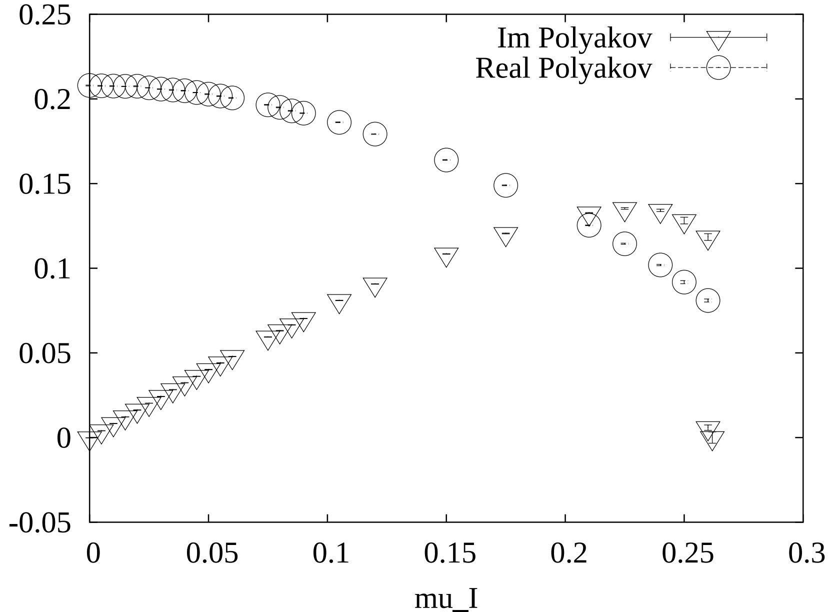
- $L =< \mathrm{Tr}P >$.

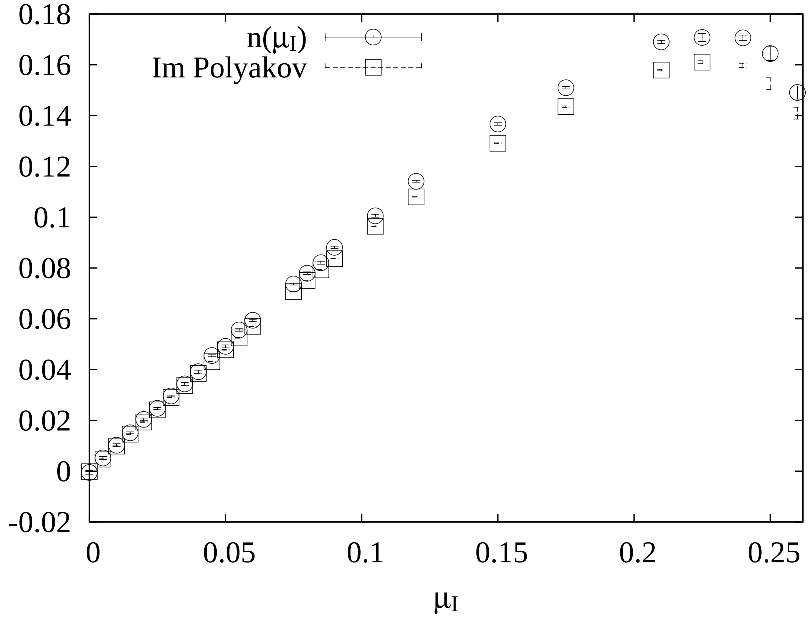
$$P(\mu) = P^\dagger(-\mu)$$

- $L =< \mathrm{Tr}P >$ and $\bar{L} =< \mathrm{Tr}P^\dagger >$ are real at real chemical potential,
- But: $L \neq \bar{L}$: Karsch et al. 1985, Dumitru and Pisarski 2005, Roessner et al. 2006, de Forcrand and Kratochvila, 2006.
- $L(\mu) \neq \bar{L}(\mu)$ when *determinant is complex* reflecting forward propagation enhanced when $\mu > 9$

Going Imaginary:

- Fate of the asymmetry which is present at real chemical potential?
- $L_{o/e}(\mu) \equiv L(\mu) \pm L(-\mu) = L(\mu) \pm \bar{L}(\mu)\,,$
- The analytic continuation of the even observable $L_e(\mu) = L(\mu) + \bar{L}(\mu)$ at imaginary chemical potential is the real part of $L(\mu_I)$
- The analytic continuation of $L_o(\mu) = L(\mu) - \bar{L}(\mu)$ is the imaginary part of $L(\mu_I)$ at imaginary μ.
- L itself has no definite μ-parity and its analytic continuation develops an imaginary part.


Im Polyakov
Real Polyakov
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
0.05
0.1
0.15
0.2
0.25
0.3
mu_I


n(μI)
Im Polyakov
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
0
0.05
0.1
0.15
0.2
0.25
μI

The theory 'knows' about the sign problem yet analytic continuation can be accomplished.

Estimate of the sign problem

$$F_{QCD} - F_{|QCD|} = \alpha_s^2 \mu^2 T^2$$

(see e.g J. Verbaarschot talk at xQCD08) : Large!

However

$$\frac{n_{QCD} - n_{|QCD|}}{n_{QCD}} = 0(\alpha_s^2) \simeq O(ln(T)^{-2})$$

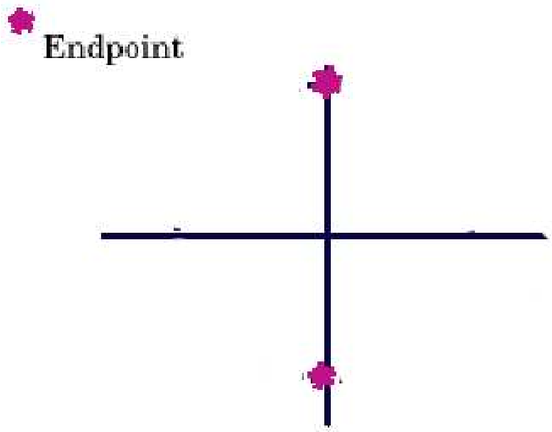
Overlap $\simeq$ Width :

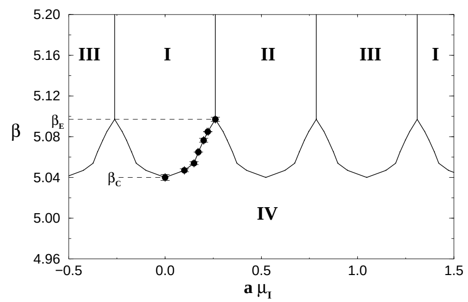
$$(\Delta N)^2 = V \chi T^2$$

Sign problem is not a problem at high T : fluctuations help

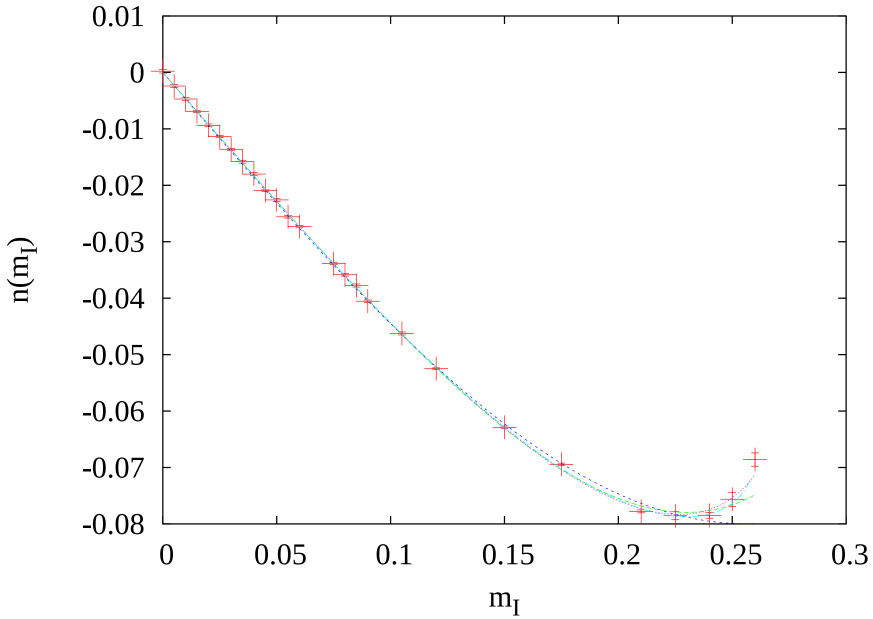
4 The Endpoint at imaginary μ

The Complex μ plane at $T = T_E^{RW}$





n(μ_I) at the endpoint



Proposed Fit

$$n(\mu_I) = A\mu_I({\mu_I^c}^2 - \mu_I^2)^\alpha \,.$$

Implying:

$$\chi_q(\mu_I) \propto \frac{1}{({\mu_I^c}^2 - \mu_I^2)^\gamma}$$

where $\gamma = 1 - \alpha$

- A fit to our entire interval with unconstrained μ_I^c gives $A = -0.94(4)$, ${\mu_I^c}^2 = 0.0804(2)$, $\alpha = 0.28(2)$ with a reduced $\tilde{\chi}^2 = 2.4$.
- If we constrain ${\mu_I^c}^2 = (\pi/12)^2$ the quality of the fits decreases giving a reduced $\tilde{\chi}^2 \simeq 12$.
- We checked the stability of these results by choosing different ranges in chemical potential, and we obtained the exponent α ranging between 0.34(8) and 0.26(3), ${\mu_I^c}^2$ between 0.078(4) and 0.091(12), with reduced $\tilde{\chi}^2$ ranging between 1.8 and 5.

CRITICAL FITS IN THE CRITICAL REGION : Further checks

- If we limit the fitting interval to $\mu_I > 0.15$, we need to add a constant to the function to approximate the regular component.

$$n(\mu_I) = A\mu_I({\mu_I^c}^2 - \mu_I^2)^\alpha \; + B,$$

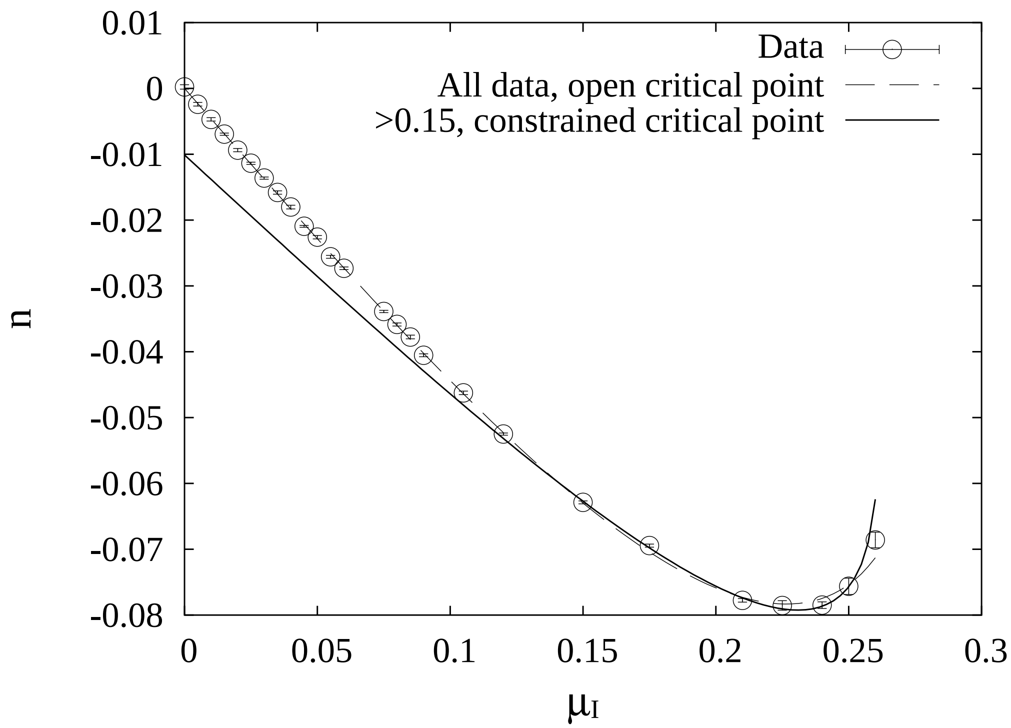
We obtain $A = -0.54(4), \alpha = 0.14(1), B = -0.010(3)$ a reduced $\tilde{\chi}^2 = 1.79$.

- We also consider a critical behaviour supplemented by a regular linear term

$$n(\mu_I) = A\mu_I({\mu_I^c}^2 - \mu_I^2)^\alpha \; + B\mu_I, .$$

This form of the regular term respects the symmetries of $n(\mu)$ hence can be used for analytic continuation, at variance with the simple modification considered above, when we supplemented $n(\mu)$ by a constant. The results for this fit are $A = -1.04(7)$, $\mu_I^2 = 0.62(7)$, $\alpha = 0.62(7)$ and $B = -0.26(4)$, with a reduced $\chi^2 = 1.74$.

RESULTS OF CRITICAL FITS FOR $n(\mu)$ AT THE CEP



FIRST OBSERVATION OF A CRITICAL BEHAVIOR IN QCD at $\mu \neq 0$

$$n(\mu_I) \propto \mu_I ({\mu_I^c}^2 - \mu_I^2)^{(\alpha)}$$

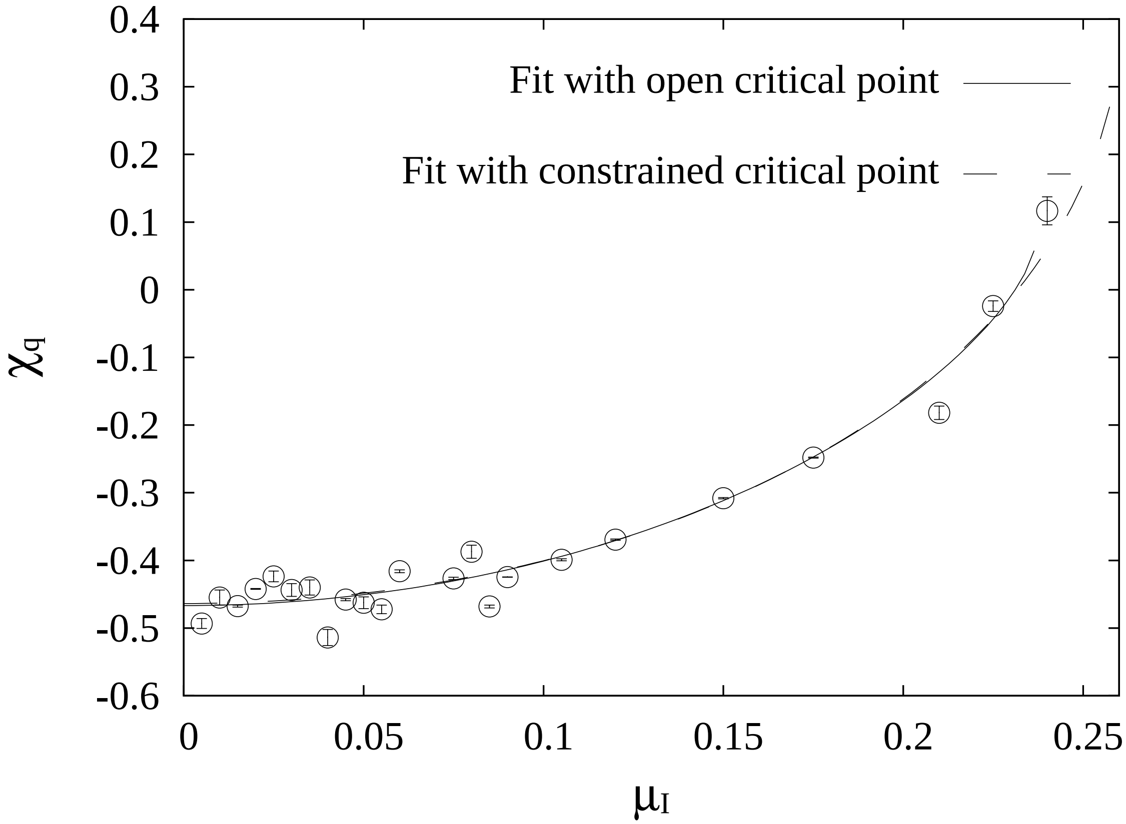
FIT I : $\alpha = 0.23(3) \gamma = 0.77(3)$

FIT II : $\alpha = 0.18(2) \gamma = 0.82(2)$

Consistent with a crossover from mean field to CEP

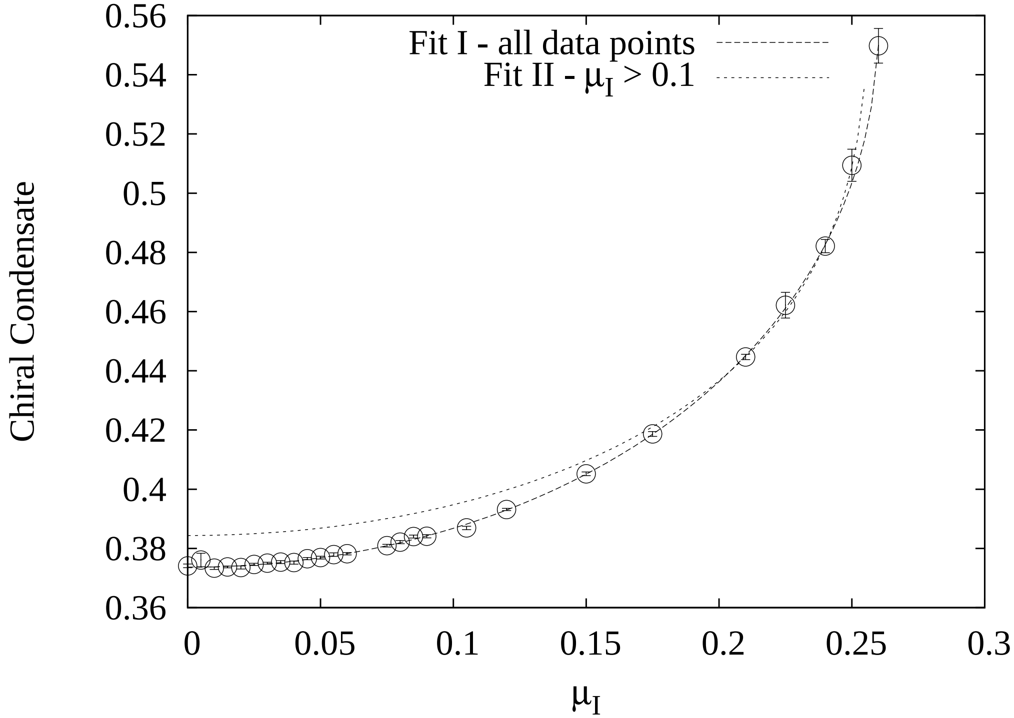
QUARK NUMBER SUSCEPTIBILITY AT A CEP

see Hatta Ikeda, Gavai Gupta for a discussion in QCD



open $\mu_I : \gamma = 0.66(16)$ constrained $\mu_I : \gamma = 0.44(22)$

CHIRAL CONDENSATE



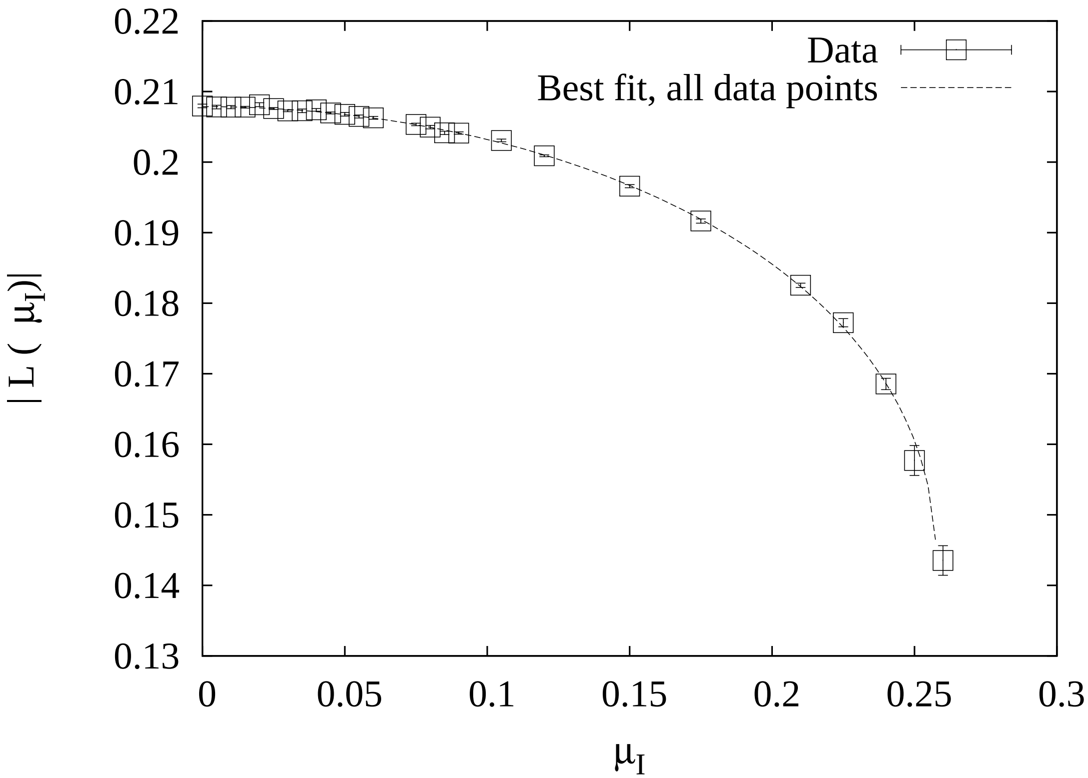
Maxwell relations:

$$\frac{\partial n(\mu,m)}{\partial m} = \frac{\partial < \bar{\psi}\psi > (\mu,m)}{\partial \mu}.$$

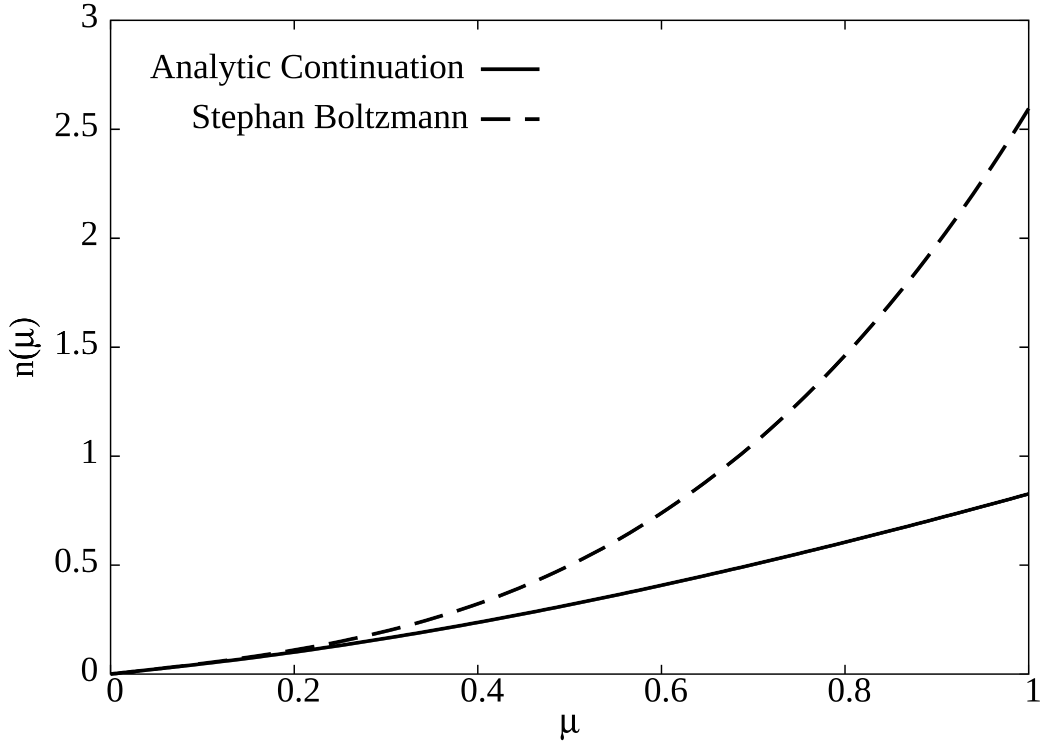
POLYAKOV LOOP

$L =< \mathrm{Tr}P >$

$L(\mu_I) \propto (\mu_I^{c\,2} - \mu_I^2)^\beta$



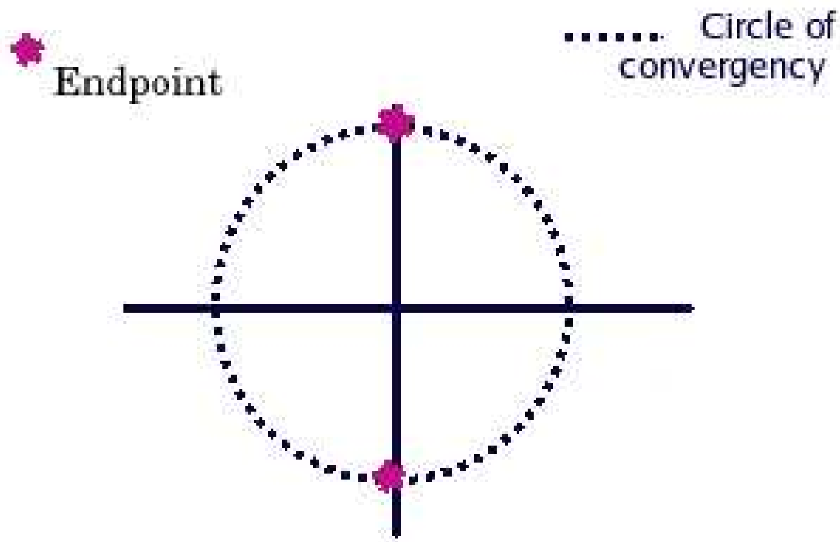
ANALYTIC CONTINUATION: EOS AT $T = 1.1Tc$



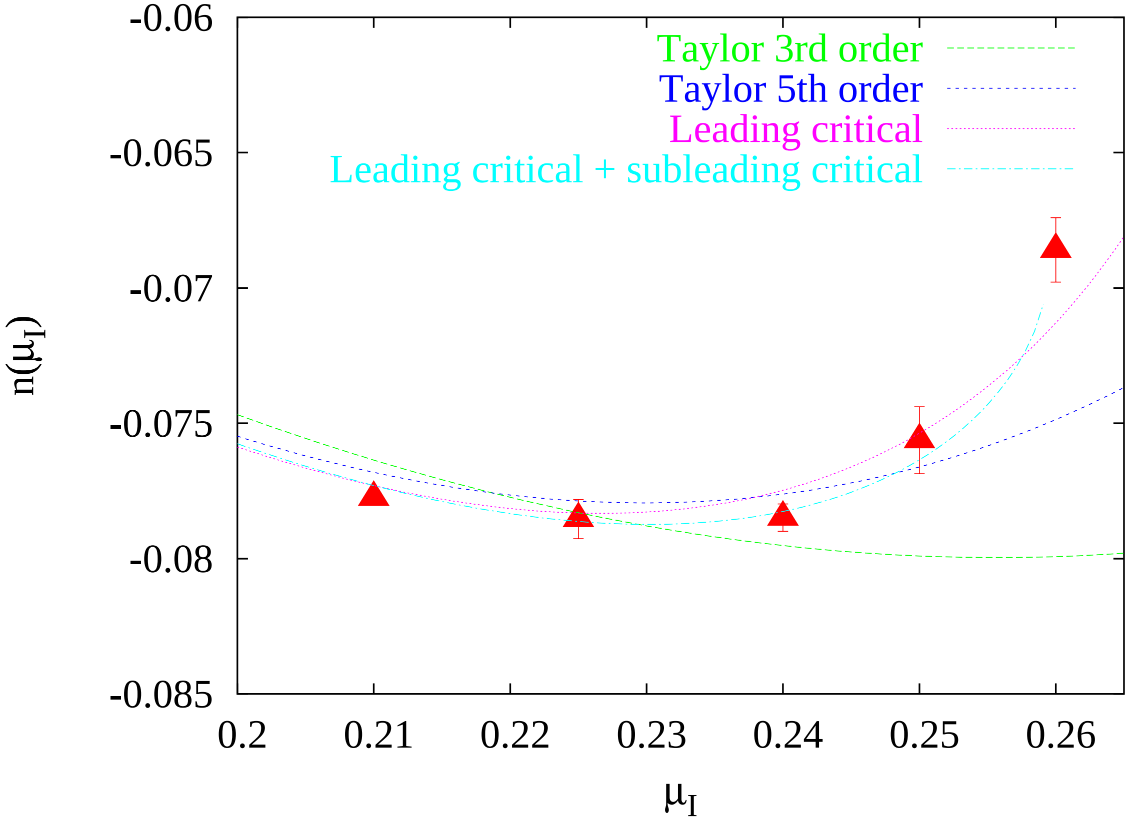
- $n(\mu)$ from analytic continuation, together with a free field behaviour.
- The fits suggest that the slower increase observed in the interacting case with respect to the free case can be described by on overall exponent smaller than one.

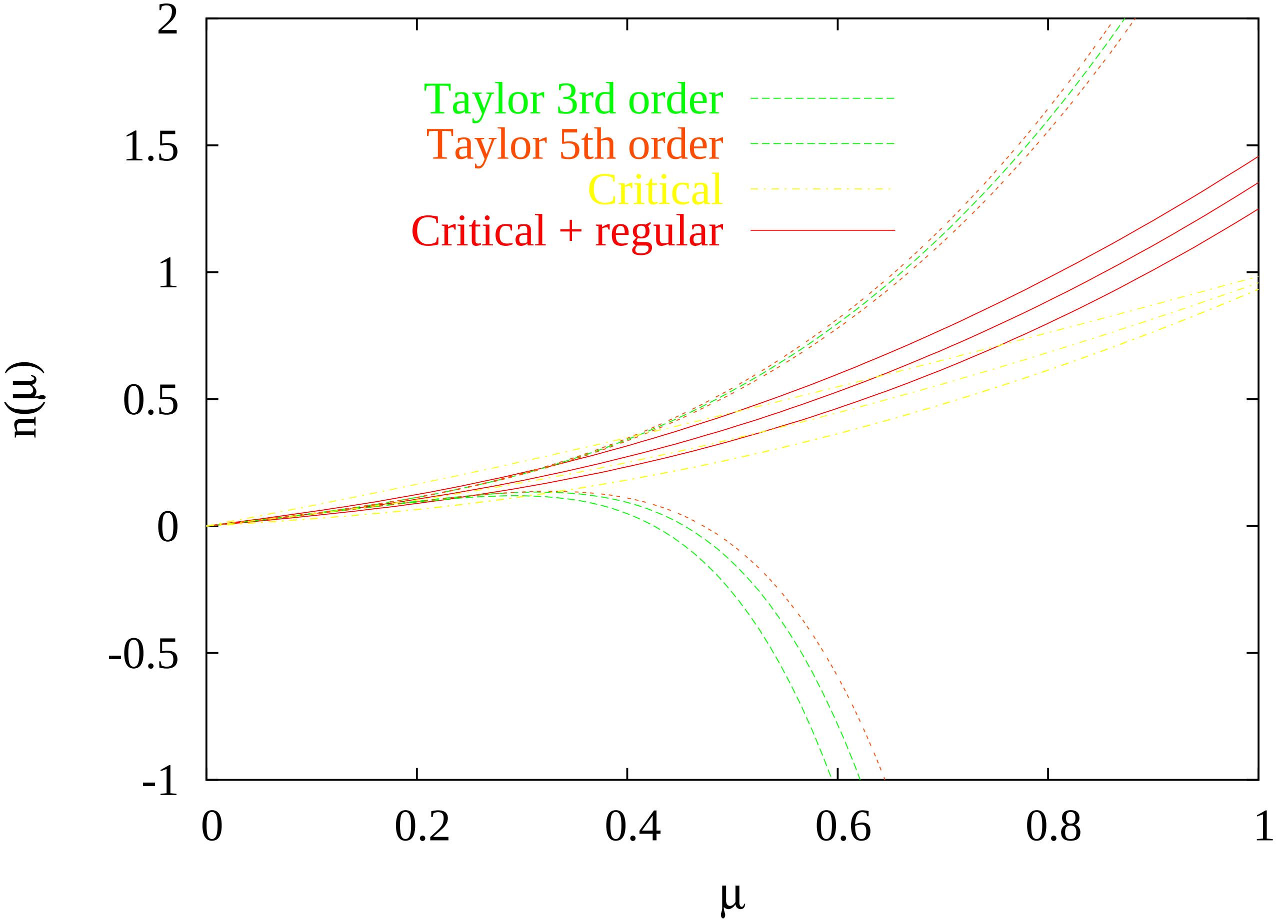
5 The Taylor expansion

NEW INGREDIENT: RADIUS OF CONVERGENCE



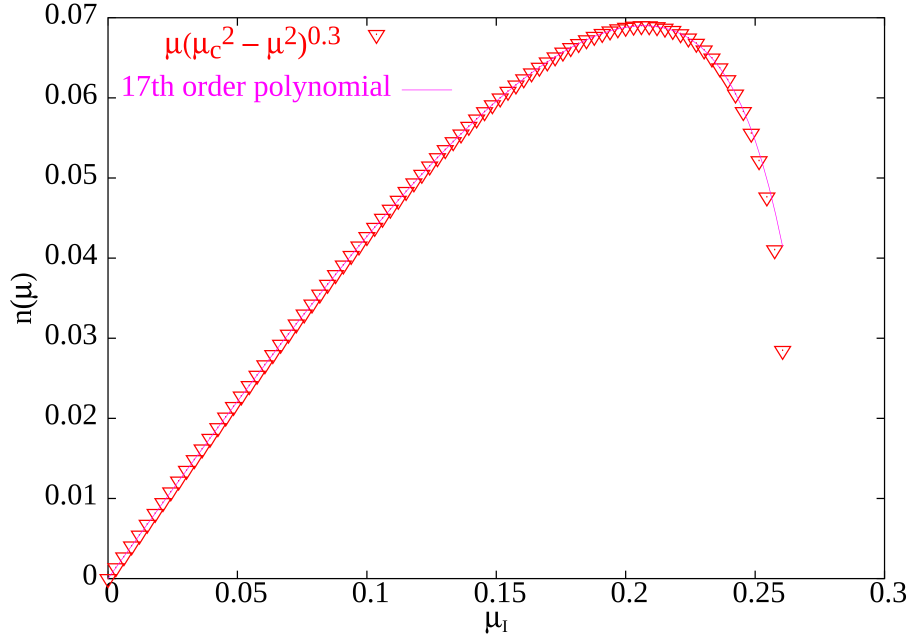
Complex μ plane

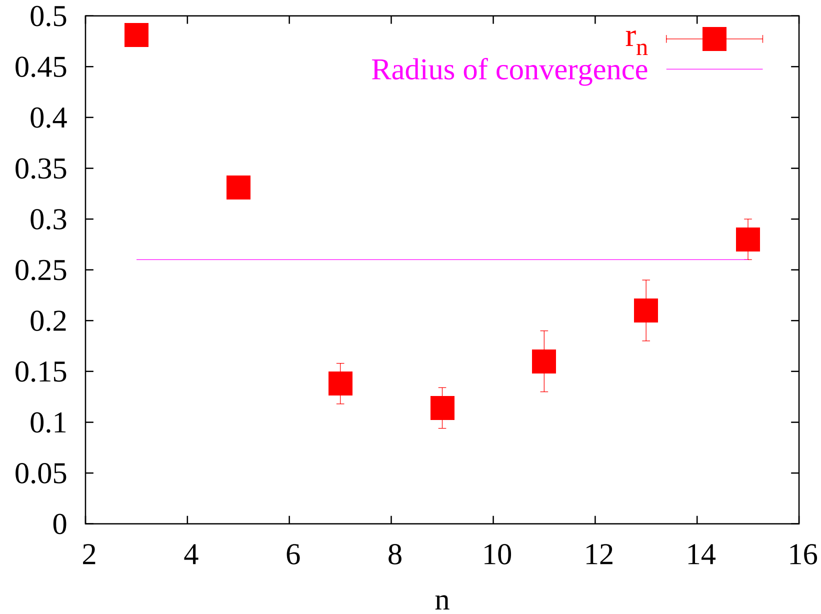

Taylor 3rd order
Taylor 5th order
Leading critical
Leading critical + subleading critical
-0.06
-0.065
-0.07
-0.075
-0.08
-0.085
0.2
0.21
0.22
0.23
0.24
0.25
0.26
$n(\mu_I)$
μ_I



FINITE RADIUS OF CONVERGENCE OF THE TAYLOR EXPANSION!

Toy Study : Taylor expanding the exact result





6 Critical fits vs HRG and Quasiparticles

.....vs HADRON RESONANCE GAS
Ejiri , Karsch, Redlich proposed the following parametrization for the contribution of the coloured states to the subtracted pressure

$$\Delta P_C = P_C(T,\mu) - P_C(T,0)$$

$$\begin{aligned} \frac{\Delta P}{T^4} &= F_q(T)(\cos(\mu/T)) - 1) + F_{qq}(T)(\cos(2\mu/T) - 1.) \\ &+ F_{qqq}(T)(\cos(3\mu/T) - 1) + +F_{qqqq}(T)(\cos(4\mu/T) - 1.) \end{aligned}$$

giving in turn:

$$\begin{aligned} n(\mu_I, T) &= F_q(T)\sin(\mu_I/T) + 2F_{qq}(T)\sin(2\mu_I/T) \\ &+ 3F_{qqq}(T)\sin(3\mu_I/T) + 4F_{qqq}(T)\sin(3\mu_I/T) \end{aligned}$$

NAIVE COMPARISON WITH HRG

- $R_B(\mu_I) = \frac{n(\mu_I)}{\sin((3)\mu_I/T)}$
  should be a constant for a simple hadron gas
- $R_q(\mu_I) = \frac{n(\mu_I)}{(3)\sin(\mu_I/T)}$
  should be a constant for a "hadron gas" made of quarks.

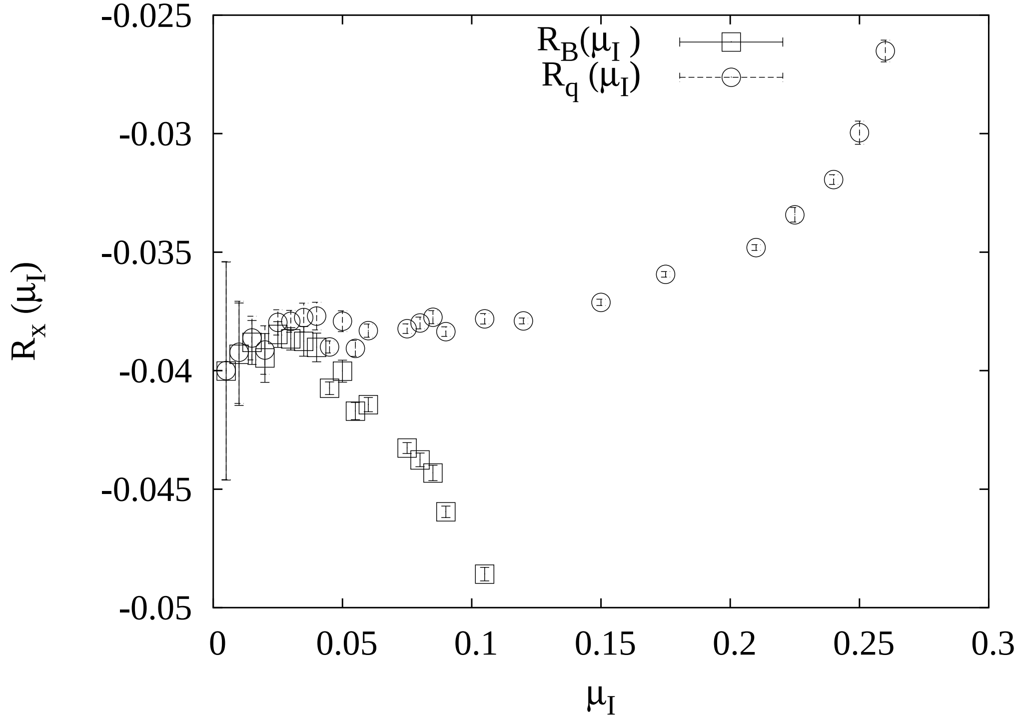


Table 1: Parameters of the Trigonometric Fits

F_1	F_2	F_3	F_4	$\chi^2/d.o.f.$
-0.110(1)	–	–	–	84
-0.071(3)	-0.023(2)	–	–	11.11
0.028(15)	- 0.114(14)	0.029(4)	–	4.18
0.43(11)	-0.55(13)	0.257(66)	-0.049(14)	2.85

SIMPLE STRATEGY : FIT TO

$$n_K(\mu_I) = \sum_{j=1}^{K} F_j \sin(j\mu_I/T)$$

.....vs. SUSCEPTIBILITIES ANALYSIS

- The susceptibilities at zero chemical potential can be easily computed and we recognise that their ratios allow the identifications of the relevant degrees of freedom. **Gavai, Gupta,2001 2008**
- These prediction for the susceptibilities ratio for quarrk,diquark, etc. around T_c was contrasted with the numerical results, finding a poor agreement . **Ejiri, Karsch, Redlich, 2007**
- The derivatives of the masses with respect to the chemical potential should depend on μ:

$$M''(T,\mu) = \frac{\partial^2 M(T,\mu)}{\partial \mu^2}(T,\mu=0)$$

  Miao, Shuryak, 2006-2008
- **When M"(T,μ) is large enough, the simple interpretation of the zero chemical potential susceptibilities as probes of particle contents has to be revised**

..VS QUASIPARTICLE MODELS

Kämpfer, Bluhm PRD2007

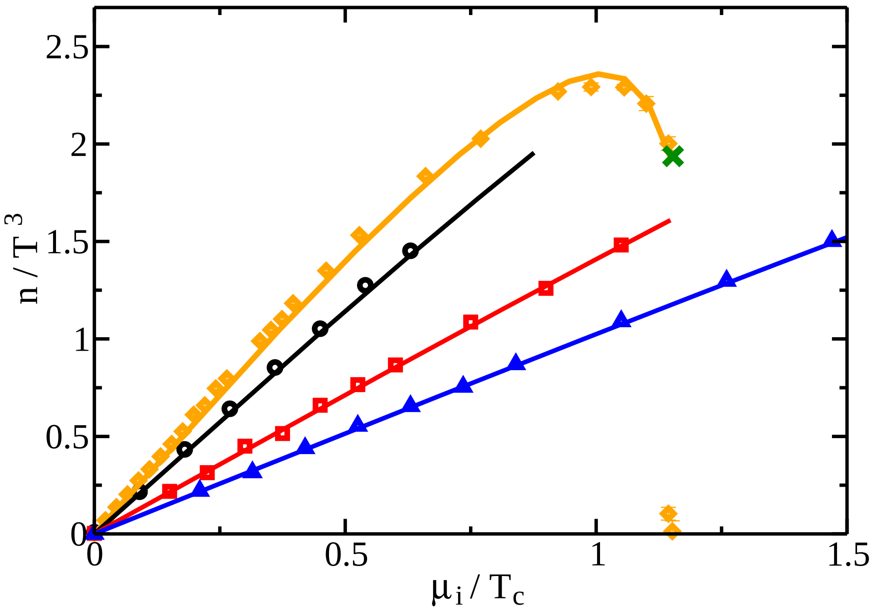
Quasiparticlemodel vs Imaginary Chemical Potential Lattice Data, and analytic continuation to Real Chemical Potential

Kämpfer, Bluhm , 2007

Crucial ingredients:

- Explicit dependence of the self-energy parts on $\mu_i = \mu_{u,d}$ and T
- Implicit dependence via the effective coupling $G^2(T, \mu_u, \mu_d)$.

$$\omega_i^2 = k^2 + m_i^2 + \Pi_i, \quad \Pi_i = \frac{1}{3}\left(T^2 + \frac{\mu_i^2}{\pi^2}\right) G^2(T, \mu_u, \mu_d) \, .$$



$T = 3.5,\ 2.5,\ 1.5,\ 1.1\ T_c$ from top to bottom

Data from M.D'Elia, F. Di Renzo, MpL 2005-2007

SUMMARY

- Endpoint at imaginary μ useful to model thermodynamics and as a test bed for methods
- Three ways of model data
  - Taylor : rigorous within circle, limited convergence
  - Pade' : incorporate singularities
  - Critical fits : incorporate singularities *and* critical exponents
- Estimate of convergence radius might require a large number of coefficients.
- Critical fits suggest EOS close to T_c

$$n(\mu) = A\,\mu\,({\mu_I}_c^2 + \mu^2)^\alpha$$

  with $\alpha \simeq 0.3$, generalizes Stefann–Boltzmann-like law.
- Critical fits complementary not alternative to other methods:
- Coherent indications that the mass spectrum is μ dependent.