Phase of the Fermion Determinant and the Phase Diagram of QCD

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References

K. Splittorff and J. J. M. Verbaarschot, Phase of the fermion determinant at nonzero chemical potential, Phys. Rev. Lett. **98**, 031601 (2007) [arXiv:hep-lat/0609076].

K. Splittorff and J.J.M. Verbaarschot, The QCD Sign Problem for Small Chemical Potential, Phys. Rev. **D75**, 116003 (2007) [arXiv:hep-ph/0702011].

K. Splittorff and J. J. M. Verbaarschot, Acta Phys. Polon. B **38**, 4123 (2007) [arXiv:0710.0704 [hep-th]].

L. Ravagli and J. J. M. Verbaarschot, Phys. Rev. D **76**, 054506 (2007) [arXiv:0704.1111 [hep-th]].

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QCD and |QCD|

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QCD and |QCD|

The two flavor QCD partition function at $\mu \neq 0$ given by

$$Z_{\rm QCD} = \langle [\det(D + m + \mu\gamma_0)]^2 \rangle.$$

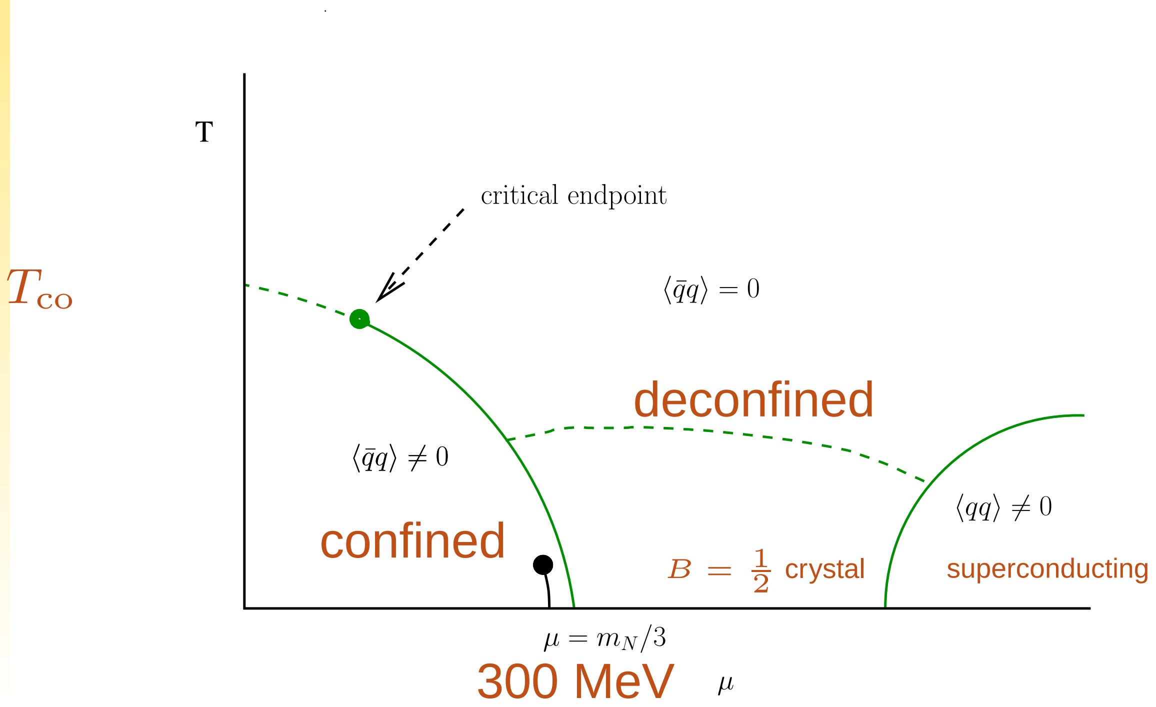
The phase quenched QCD partition function only differs from this by the phase of the fermion determinant

$$Z_{|\rm QCD|} \langle |\det(D + m + \mu\gamma_0)|^2 \rangle = \langle \det(D + m + \mu\gamma_0) \det(D + m - \mu\gamma_0) \rangle.$$

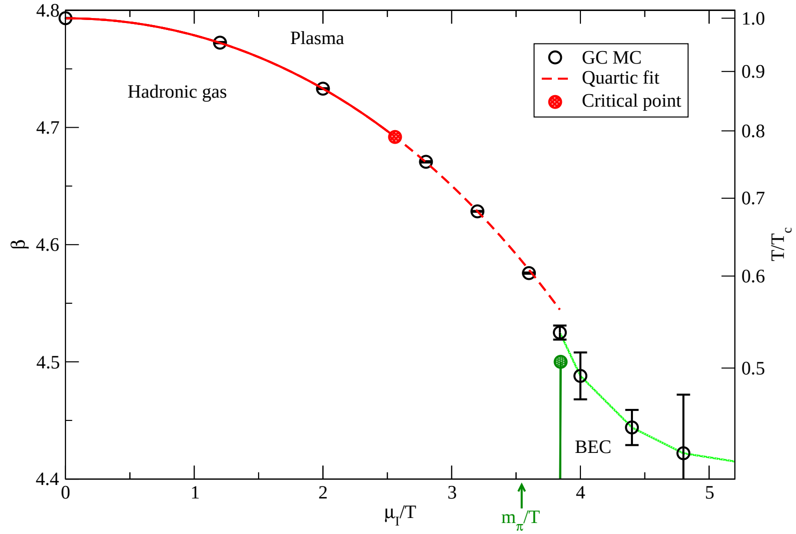
Because of this it can also be interpreted as QCD at isospin chemical potential $\mu_I = \mu$.

$Z_{\rm QCD}$ cannot be simulated by probabilistic methods but standard lattice simulations are feasible for $Z_{|\rm QCD|}$.

Phase Diagram of QCD and |QCD|



Schematic QCD phase diagram.

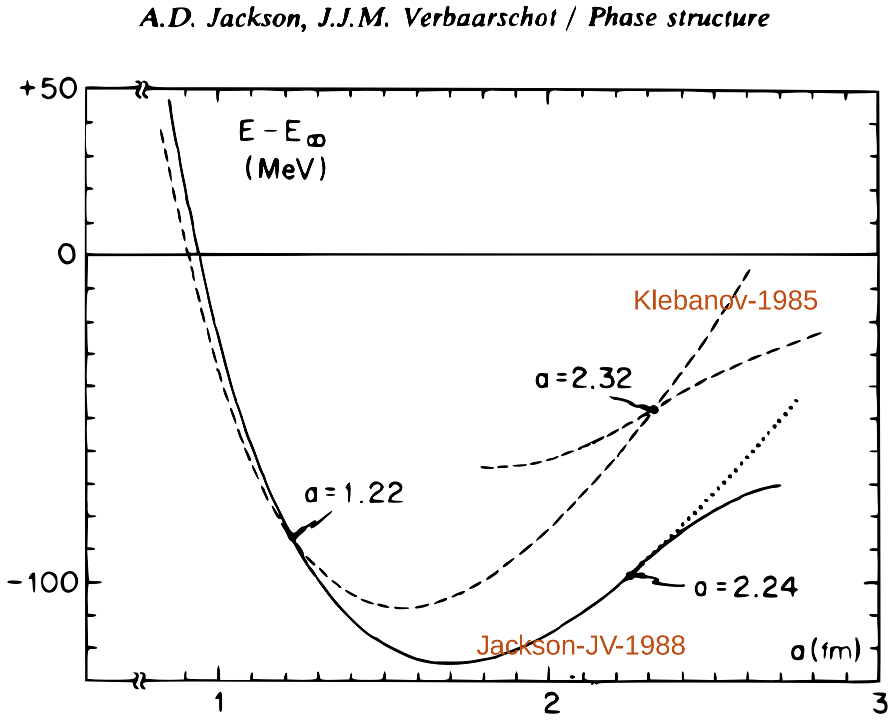


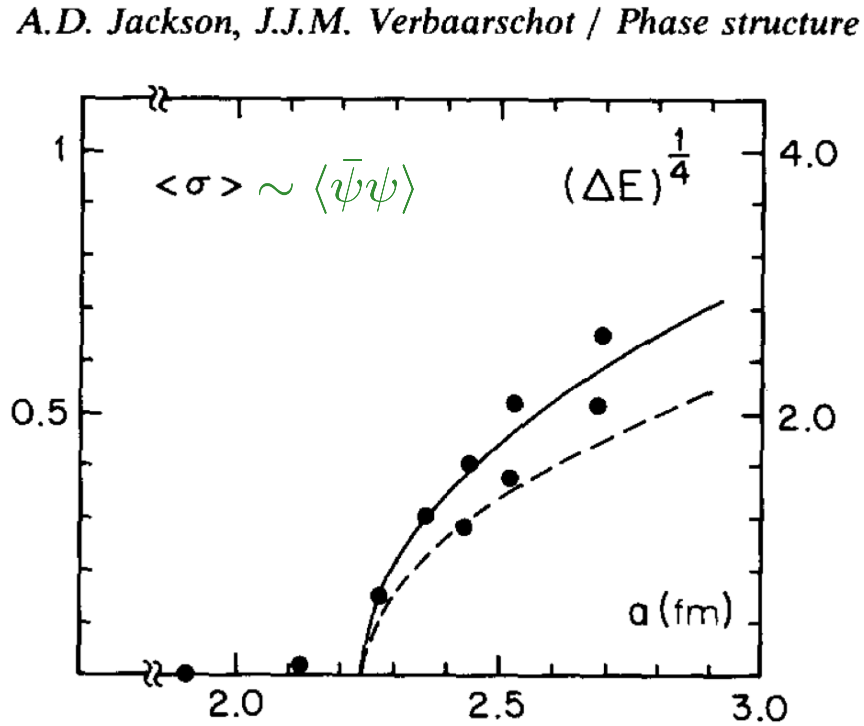
Phase diagram of phase quenched QCD (de Forcrand-Stephanov-Wenger-2007). Agrees with earlier work by Kogut and Sinclair).

$Z_{|\mathrm{QCD}|}$ has a phase transition at $\mu = m_\pi/2$ so that the free energies of the two theories are completely different.

An nonzero temperature the free energies are different for any nonzero value of the chemical potential.

Skyrme Crystal



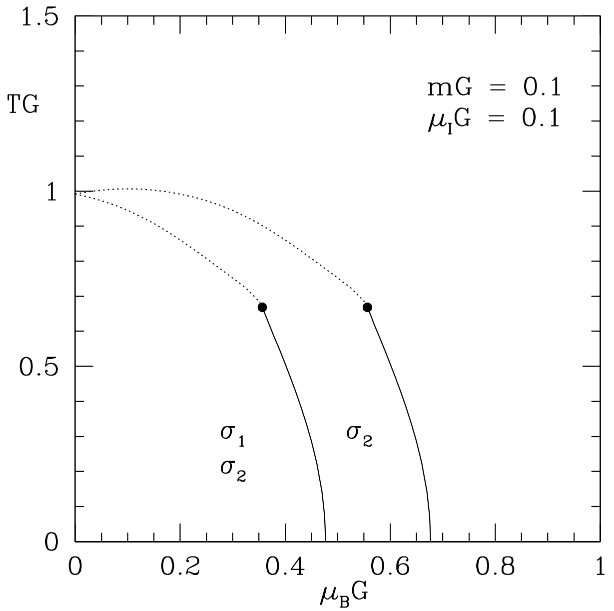


Chiral symmetry restoration in a Skyrme crystal.

Jackson-JV-1988

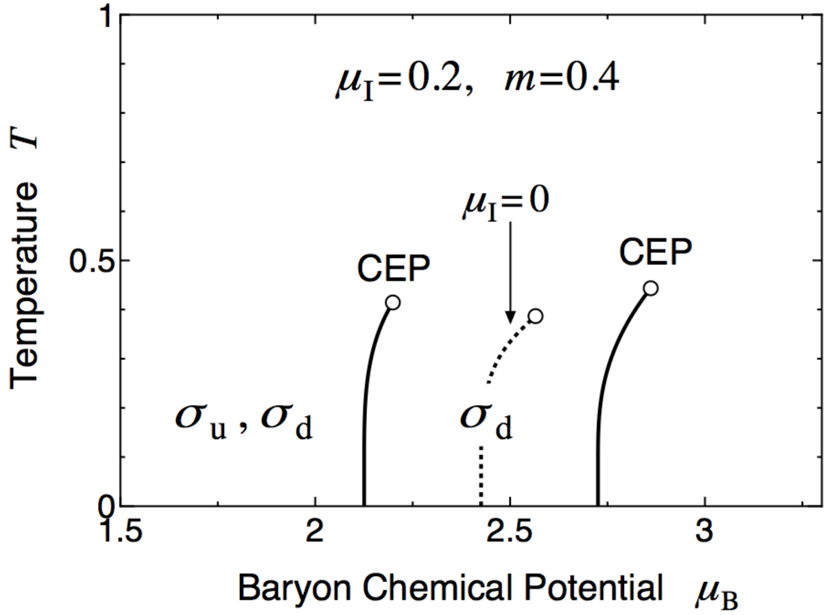
Quarkyonic phase. McLerran-Pisarski-2007

Phase Diagram at Nonzero Isospin Chemical Potential



Phase Diagram at Nonzero Isospin Chemical Potential.

Klein-Toublan-JV-2003



Phase Diagram of Strong Coupling QCD at Nonzero Isospin Chemical Potential Nishida-2003

Critized by Van Oertel-et all, Schaefer, ...

Severity of Sign Problem for $\mu \neq 0$

Because the Dirac operator at nonzero μ is nonhermitean, the fermion determinant is complex

$$\det(D + \mu\gamma_0 + m) = e^{i\theta} |\det(D + \mu\gamma_0 + m)|.$$

If the average phase factor vanishes in the thermodynamic limit, Monte-Carlo simulations are not possible (sign problem).

The severity of the sign problem can be measured by the ratio

$$\langle e^{2i\theta} \rangle \equiv \frac{\langle \det^2(D + m + \mu\gamma_0) \rangle}{\langle |\det(D + m + \mu\gamma_0)|^2 \rangle} \sim e^{-V(F_{N_f=2} - F_{pq})}$$

full QCD
partition function

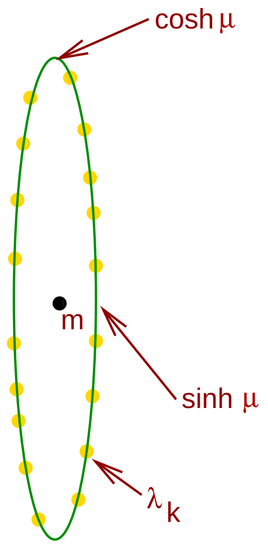
phase quenched
partition function

QCD in one Dimension

Dirac operator:

$$D = \begin{pmatrix} mI & e^{\mu} & \dots & e^{-\mu}U^{\dagger} \\ -e^{-\mu} & mI & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & mI & e^{\mu} \\ -e^{\mu}U/2 & \cdots & -e^{-\mu} & mI \end{pmatrix}$$

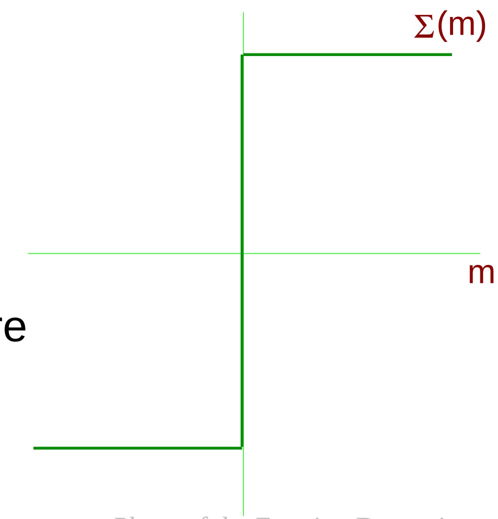
$U(N_c)$
matrix



Dirac spectrum of 1d QCD

$$\Sigma(m) = \frac{\left\langle \sum_k \frac{1}{\lambda_k + m} \prod_k (\lambda_k + m) \right\rangle}{\langle \prod_k (\lambda_k + m) \rangle}$$

determinant with
a complex phase



The chiral condensate has a discontinuity in region where there are no eigenvalues

Ravagli-JV-2007

Overlap Problem

It is possible to put the phase factor in the observable and use gauge field configurations generated by $Z_{|QCD|}$ (known as reweighting).

This might introduce the overlap problem, namely that observables for the ensemble that is generated seem to converge to the wrong value.

For example, the chiral condensate is close to the phase quenched value for each separate gauge field configuration

$$\frac{1}{V}\mathrm{Tr}\frac{1}{D+\mu\gamma_0+m} = \Sigma_{|\mathrm{QCD}|} + \delta\Sigma,$$

with $\delta\Sigma \ll \Sigma_{|\mathrm{QCD}|}$ for a *finite* ensemble.

The true value of the condensate is reached from rare but very large fluctuations, many orders of magnitude bigger than a typical value.

The deep reason for this that $\langle e^{2i\theta}\rangle$ is the ratio of two partition functions, and depends exponentially on the volume.

Teflon Plated Observables

These are observables that have no correlations with the phase factor,

$$\underbrace{\langle \mathcal{O} e^{2i\theta} \rangle_{|QCD|}} = \langle \mathcal{O} \rangle_{|QCD|} \underbrace{\langle e^{2i\theta} \rangle_{|QCD|}}.$$

Then

$$\langle \mathcal{O} \rangle_{QCD} = \langle \mathcal{O} \rangle_{|QCD|}.$$

Examples:

- √ Chiral condensate for $\mu < m_\pi/2$ and very low temperatures.
- √ Baryon density for $T \gtrsim T_{\rm co}$.
- √ Baryon density at very low temperatures and $\mu < m_\pi/2$.
- √ $T_{\rm co}(\mu)$ Allton-et al-2002

There is no sign problem or overlap problem but do we learn anything about QCD at nonzero chemical potential?

Is the Critical Endpoint Teflon Plated?

- If $T_{\rm co}(\mu)$ is teflon plated, it means that the convergence radius cannot be larger than $\mu = m_\pi/2$.
- Lattice simulations indicate that reweighting works in the crossover region. Fodor-Katz-2004
- Lattice simulations find a critical end point close to $\mu = m_\pi/2$ Fodor-Katz-2002, Splittorff-2006.

Next we will analyze the difference between QCD and |QCD| in terms of Dirac spectra.

II. Phases of QCD and the Dirac Spectrum

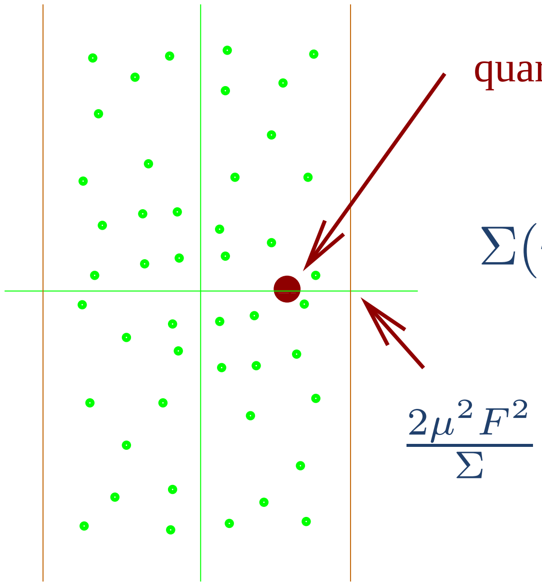
Spectral Flow with Temperature and Chemical Potential

Dirac Spectrum of Full QCD

Phase Diagram of Dirac Spectrum

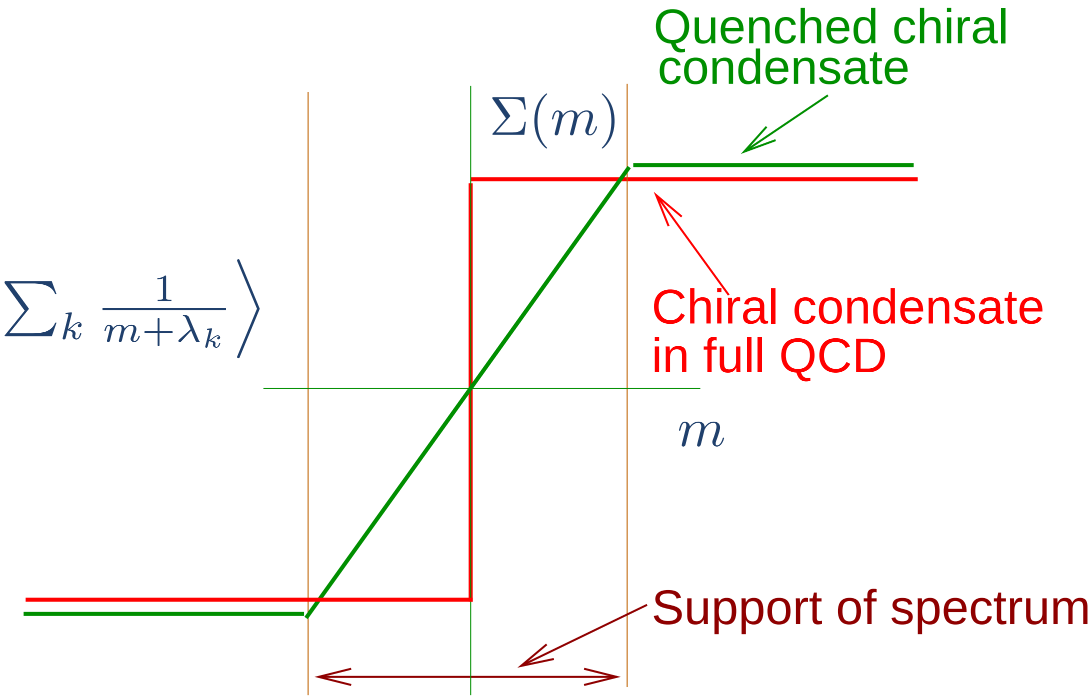
Scenarios for First Order Phase Transition

Chiral Symmetry Breaking at $\mu \neq 0$



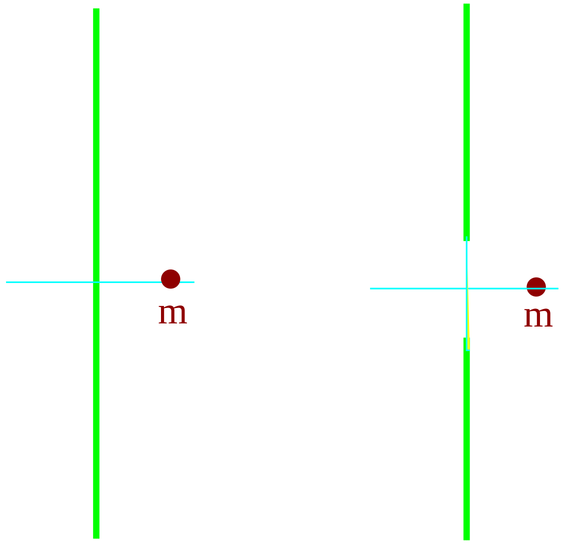
Scatter plot of Dirac eigenvalues

$$\Sigma(m) = \frac{1}{V} \left\langle \sum_k \frac{1}{m+\lambda_k} \right\rangle$$

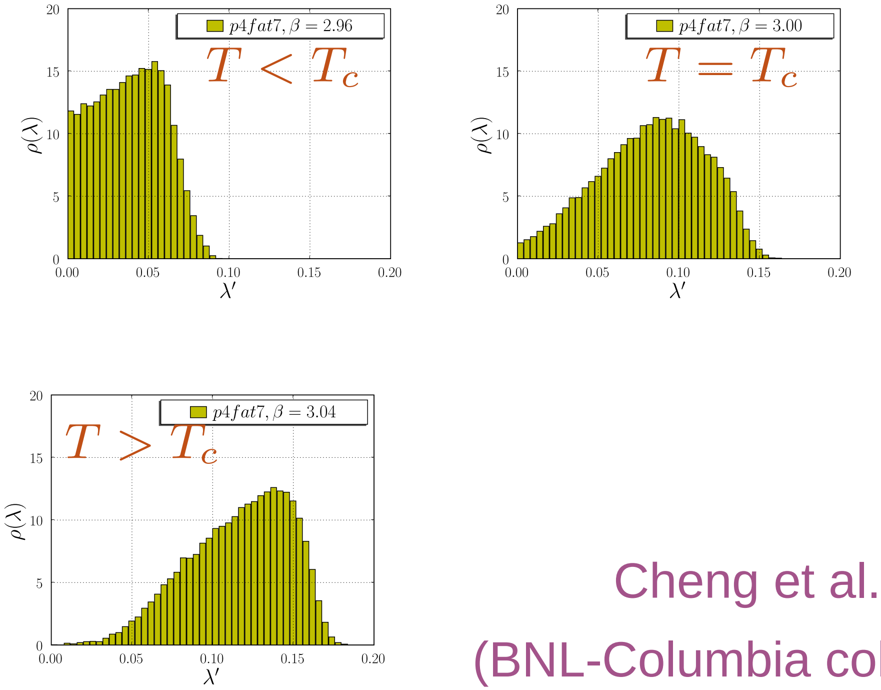


QCD and |QCD| are very different if the quark mass is inside the domain of Dirac eigenvalues: The chiral condensate in |QCD| vanishes for $m = 0$ but has a discontinuity in the real world.

Spectral Flow with Increasing Temperature



$T < T_c$ $T > T_c$



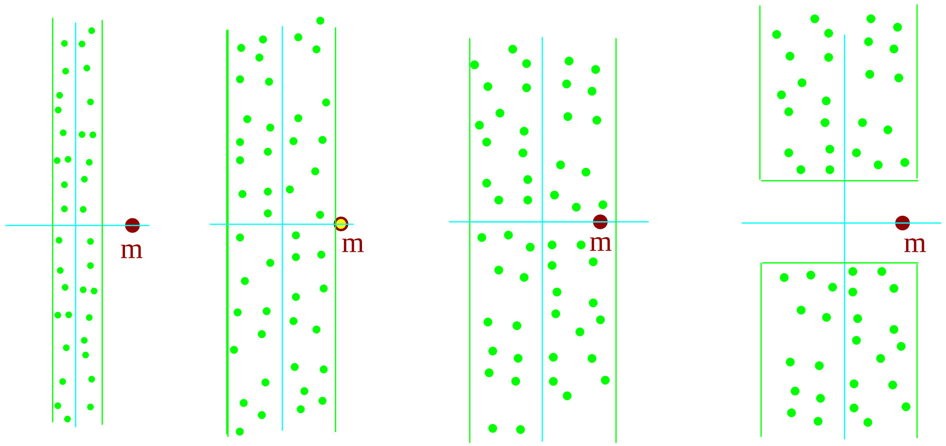
Cheng et al. 2006
(BNL-Columbia collaboration)

For $T > T_c$ the discontinuity in the chiral condensate is absent because of the formation of a gap: chiral symmetry is restored.

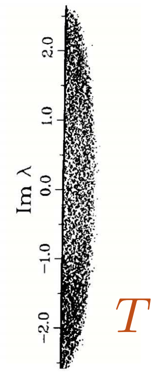
The phase transition is controlled by the flow of the small eigenvalues.

In fact, the critical temperature can be determined from the fluctuations of the smallest nonzero eigenvalue.

Spectral Flow at $\mu \neq 0$



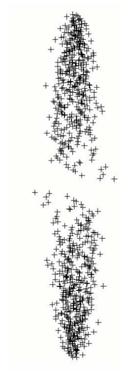
$T < T_c$ $\mu < \frac{m_\pi}{2}$ | $T < T_c$ $\mu = \frac{m_\pi}{2}$ | $T < T_c$ $\mu > \frac{m_\pi}{2}$ | $T > T_c$ $\mu > \frac{m_\pi}{2}$



$T < T_c$

$4^3 \times 8$ lattice

Barbour et al. 1986



$T > T_c$

4^4 lattice

Muroya et al. 2005

For quenched of phase quenched QCD a phase transition occurs when the mass hits the boundary of the spectrum.

For $T > T_c$ the quark mass is outside the domain of eigenvalues.

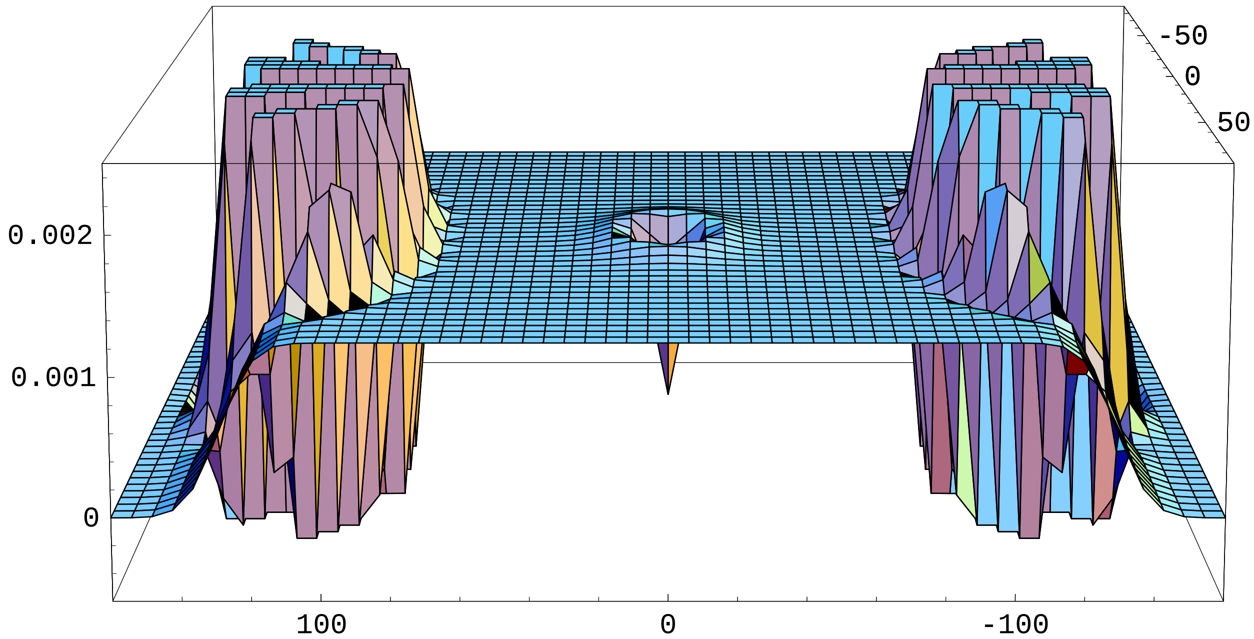
Spectral Density of Full QCD

The spectral density at nonzero chemical potential is defined by

$$\rho(\lambda) = \langle \sum_k \delta(\lambda - \lambda_k) \rangle.$$

- The average contains a complex determinant and therefore the spectral “density” is in general complex.
- At nonzero chemical potential the quenched or phase quenched chiral condensate vanishes even at low temperatures.
- The obtain a nonvanishing result the spectral “density” in full QCD has to be drastically different.
- It turns out that the spectral density contains a region with complex oscillations with a period $\sim 1/V$ and an amplitude that grows exponentially with the volume.

QCD at $\mu \neq 0$



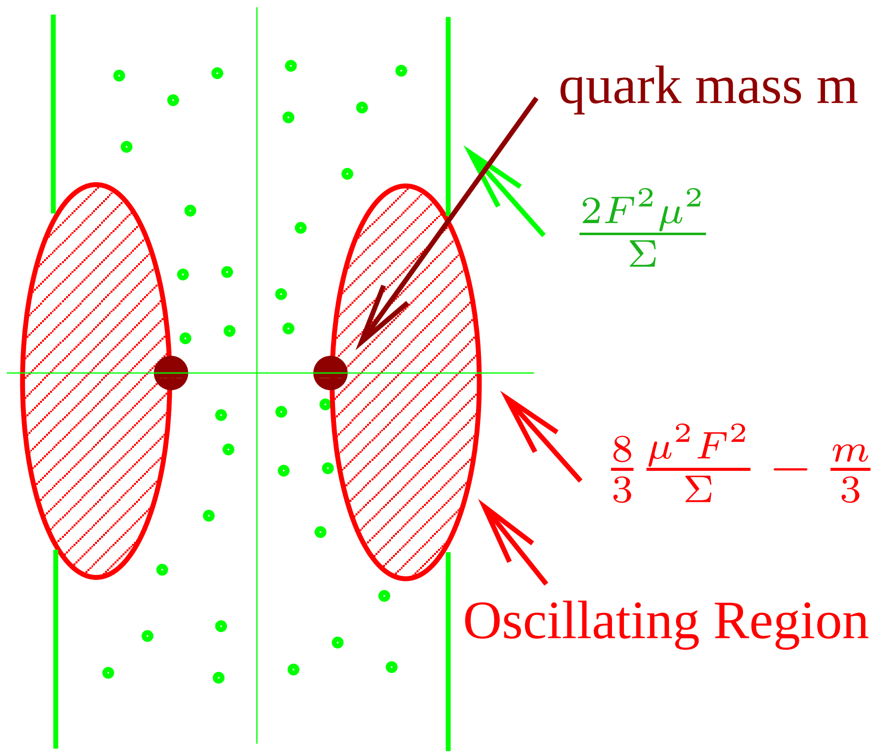
Real part of the spectral density for QCD with one flavor at nonzero chemical potential.

The oscillatory region is responsible for the discontinuity in the chiral condensate.

Osborn-Splittorf-JV-2005/2008, Osborn-2004

$\longrightarrow$

Phase Digram of Dirac Spectrum



Dirac spectrum for Full QCD.

The dotted region is in a pion condensed phase.
The dashed region is in a kaon condensed phase.
The remainder of the complex phase is in the normal phase.

Osborn-Splittorff-JV-2005/2008, Osborn-2004

Comments

- √ The oscillatory region is responsible for the discontinuity in the chiral condensate.
- √ This implies that the oscillatory region has to vanish for $T > T_c$
- √ The oscillatory region is absent for $\mu < m_\pi/2$.
- √ For $\mu > m_\pi/2$ we have two independent mechanisms to restore chiral symmetry:

  A gap develops in the Dirac spectrum.
  The oscillatory region shrinks to zero.

Scenario for First Order Phase Transition

- Spectral flow tends to be continuous and chirally induced phase transitions tend to be continuous.
- Random matrix theory with dynamical quarks has a first order phase transition at $\mu = \mu_c$ and low temperatures (Stephanov-1996). For $\mu < \mu_c$ the oscillatory part of the spectrum is responsible for the chiral condensate (Osborn-Splittorff-JV-2005).
- Because of exponentially large cancellations chiral symmetry restoration without deconfinement can be naturally induced by shrinking the oscillatory region to zero.
- The first order transition in phase quenched QCD observed in De Forcrand-Stephanov-Wenger-2007 for small chemical potentials is induced by a deconfinement transition.

Prediction for Critical Endpoint

- In other words a first order transition only occurs at chemical potentials for which we have an oscillatory region.
- If this scenario is correct it implies that $\mu_E > m_\pi/2$.
- This prediction is in agreement with lattice QCD simulations which show that the chemical potential of the critical endpoint is only sligtly larger than $m_\pi/2$. Fodor-Katz-2002/2007, Splittorff-2006

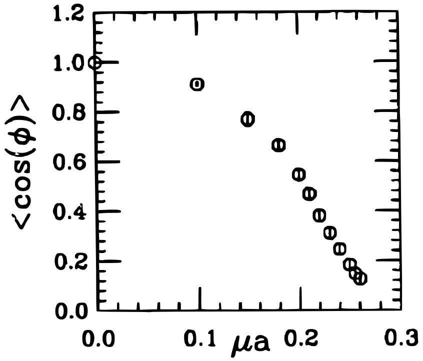
III. Phase of the Fermion Determinant

Phase and Dirac Eigenvalues

Sign Problem in QCD at $\mu \neq 0$

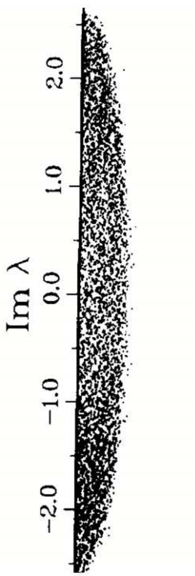
Phase Factor and Partition Functions

Phase Factor and Dirac Eigenvalues

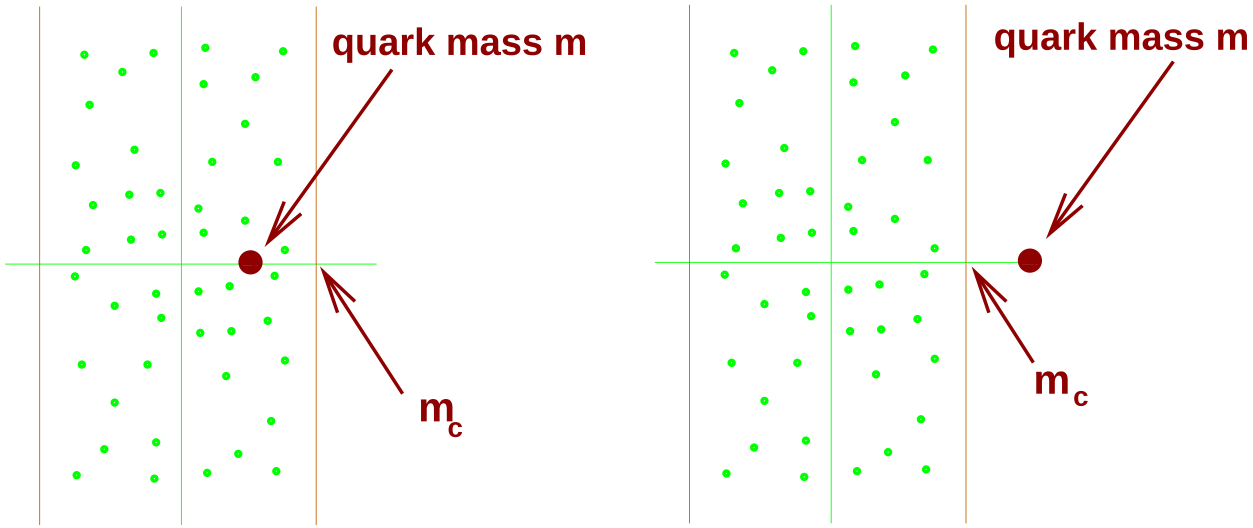


Toussaint-1990

$$\det(D + \underbrace{m}_{\prod_k(\lambda_k + m)} + \mu\gamma_0) = \underbrace{e^{i\theta}}_{\text{phase factor}} |\det(D + m + \mu\gamma_0)|$$



Barbour et al. 1986



Scatter plot of Dirac eigenvalues

$$m < m_c \qquad \text{then} \qquad \langle e^{i\theta} \rangle \sim 0$$

Sign Problem in QCD at $\mu \neq 0$

Seriousness of the sign problem: *No problem* if $\langle \theta^2 \rangle^{1/2} < \frac{\pi}{2}$
Mild if $\langle \theta^2 \rangle^{1/2} > \frac{\pi}{2}$ but $\langle \theta^2 \rangle \sim V^0$,
Serious if $\langle \theta^2 \rangle \sim V$,
$\langle e^{i2\theta} \rangle \sim e^{-VF}$.

We have seen that the latter possibility arises naturally.

$$\langle e^{2i\theta} \rangle_{|\mathrm{QCD}|} = \frac{\langle (\det(D+m+\mu\gamma_0))^2 \rangle}{\langle |\det(D+m+\mu\gamma_0)|^2 \rangle} \equiv \frac{Z^{\mathrm{QCD}}_{N_f=2}(\mu)}{Z^{|\mathrm{QCD}|}_{N_f=2}(\mu)} = \frac{Z^{\mathrm{QCD}}_{N_f=2}(\mu)}{Z^{\mathrm{QCD}}_{N_f=2}(\mu_I=\mu)}$$
$$\sim e^{-V(F_{\mathrm{QCD}}-F_{|\mathrm{QCD}|})}.$$

If $F_{\mathrm{QCD}} > F_{|\mathrm{QCD}|}$ the sign problem is exponentially hard (NP-hard) with computational complexity $\sim e^{2V(F_{\mathrm{QCD}}-F_{|\mathrm{QCD}|}}$. Troyer-Wiese-2005

Sign problem remains for $N_c \to \infty$:

$$F_{\mathrm{QCD}}(\mu) = F_{|\mathrm{QCD}|}(\mu)[1+O(\frac{1}{N_c})], \qquad (\mathrm{Cohen}-2004),$$

Remarks

- Eigenvalues are distributed more or less homogeneously inside a strip.
- The strip has a hard edge.
- Convergence of the average phase factor. What is the asymptotic p dependence of the ratio

$$\frac{\langle \prod_{k=-p}^{p} (\lambda_k^{\rm QCD} + m) \rangle}{\langle \prod_{k=-p}^{p} (\lambda_k^{|\rm QCD|} + m) \rangle} \quad ?$$

- If the chemical potential is in the microscopic domain (i.e. $\mu^2 F_\pi^2 V = \text{fixed}$ for $V \to \infty$), this ratio is determined by eigenvalues in the microscopic domain (i.e., $\lambda_k \ll 1/F_\pi \sqrt{V}$).
- Random matrix theory suggest that for finite μ the convergence might be as slow as $O(\sqrt{N/p})$.
- The phase factor is essential for physical observables.

IV. Phase Factor in Chiral Perturbation Theory

One Loop Result

Comparison with Lattice Results

Probability Distribution of the Phase

One Loop Chiral Perturbation Theory

The chiral Lagrangian depends on the the isospin chemical potential but not on the the quark number chemical potential.

To one loop order we find:

$$\langle \det{}^2(D+m+\mu\gamma_0)\rangle \quad \sim \quad e^{-VF^{(0)}_{N_f=2}} \prod_k \prod_p \frac{1}{\sqrt{m_k^2+\vec{p}^2+p_0^2}},$$

$$\langle |\det(D+m+\mu\gamma_0)|^2\rangle \quad \sim \quad e^{-VF^{(0)}_{pq}} \prod_k \prod_p \frac{1}{\sqrt{m_k^2+\vec{p}^2+(p_0-2i\mu)^2}}.$$

Only charged Goldstone bosons contribute to the ratio of the two partition functions. This results in

$$\langle e^{2i\theta}\rangle_{\rm pq} \quad = \quad \frac{(m_\pi-2\mu)(m_\pi+2\mu)}{m_\pi^2} e^{h(m_\pi^2L^2,\mu^2L^2)},$$

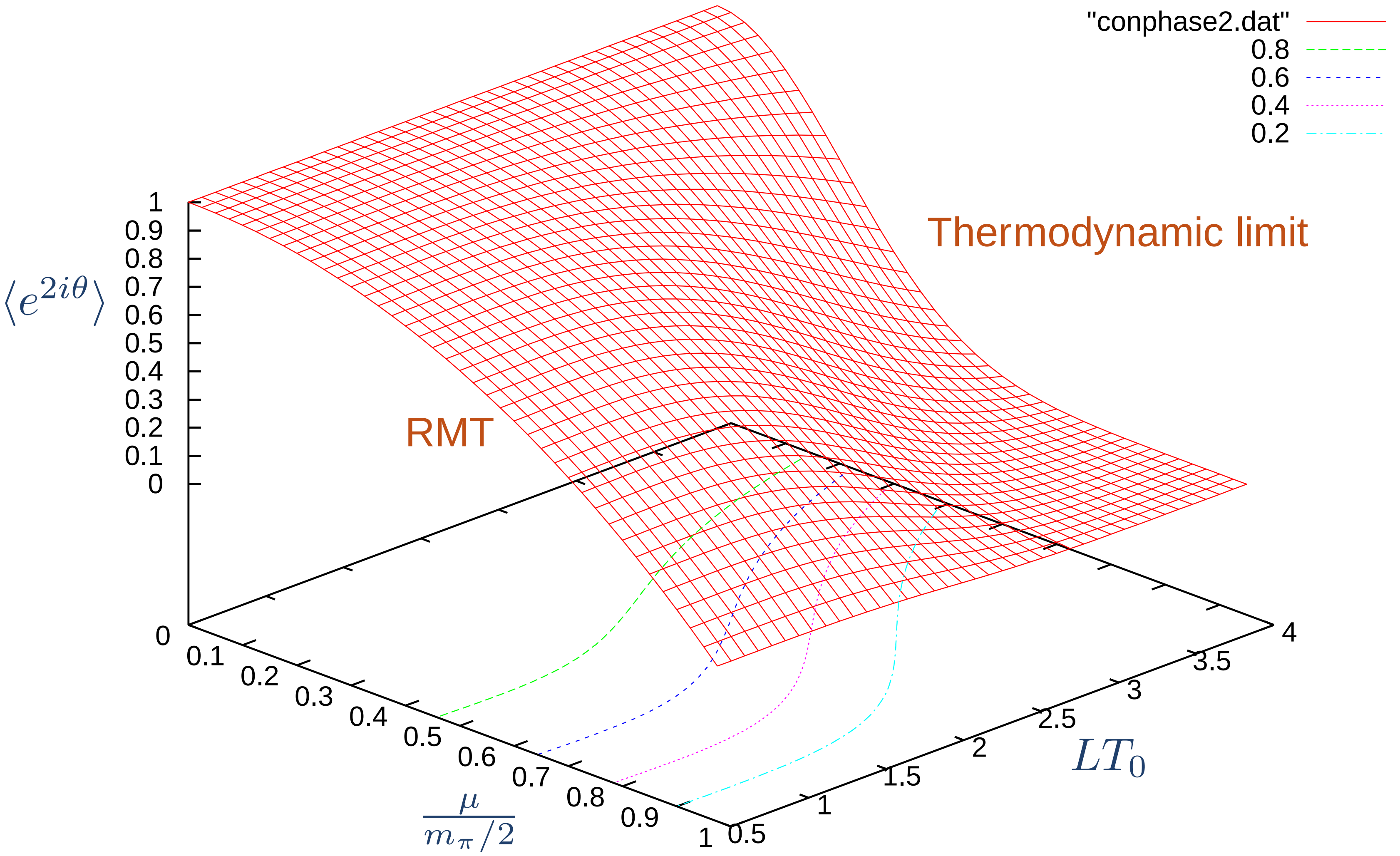
with h a finite function.

Splittorff-JV-2007

zero momentum contribution

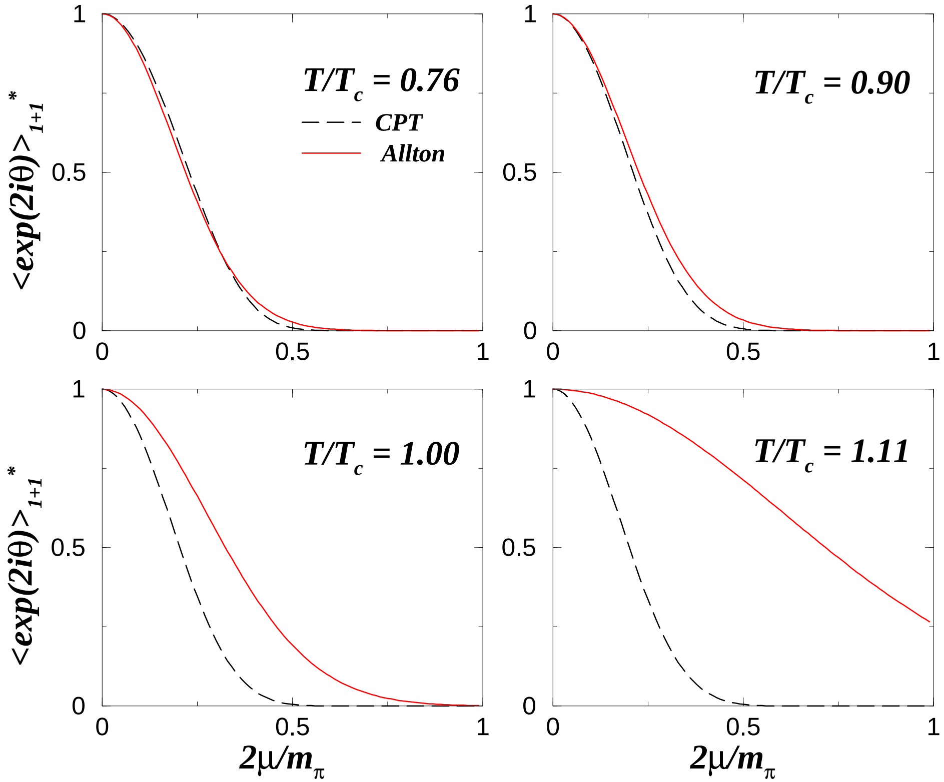
can be derived from random matrix theory

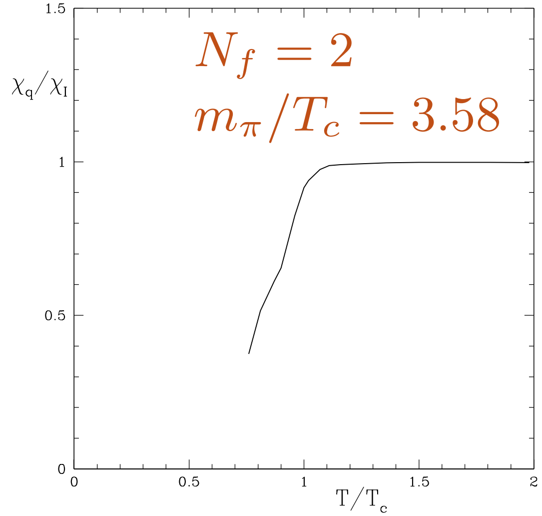
One-Loop Result



Splittorff-JV-2007

Comparison with Lattice Simulations





Ratio of quark and isospin susceptibility (χ_q/χ_I) to second order in μ (data: Allton et al. 2005)

Average phase factor in lattice QCD using the lowest order Taylor expansion (Allton-et-al.-2005) compared to one loop chiral perturbation theory in a box equal to the size of the lattice.

$$\langle e^{2i\theta}\rangle_{1+1^*} = \frac{Z_{\mathrm{QCD}}(\mu)}{Z_{|\mathrm{QCD}|}(\mu)}$$
$$\sim e^{V\mu^2(\chi_q-\chi_I)}.$$

Probability Distribution of the Phase

The density of the phase angle is defined by

$$\rho(\phi) = \langle \delta(\phi - \overbrace{\mathrm{Im}\log\det(D + m + \mu\gamma_0)}^{\theta}) \rangle_{N_f}$$

Notice that $\phi \in \langle -\infty, \infty \rangle$.

- According to the Central Limit Theorem we expect that $\rho(\phi)$ is a Gaussian. Ejiri-2007
- If the average is over dynamical quarks, the phase density is complex,

$$\begin{aligned}\langle \delta(\phi - \theta) e^{iN_f\theta} &|{\det}^{N_f}(D + m + \mu\gamma_0)|\rangle \\ &= e^{iN_f\phi} \langle \delta(\phi - \theta) |{\det}^{N_f}(D + m + \mu\gamma_0)|\rangle \ .\end{aligned}$$

- Observables are determined by correlations with the phase of the fermion determinant. Knowing the Gaussian distribution is clearly not sufficient except for teflon plated observables.

Derivation of the Phase Density

$$\begin{aligned}
\rho_{N_f}(\phi) &= \langle \delta(\phi - \mathrm{Im}\log\det(D + m + \mu\gamma_0)) \rangle_{N_f} \\
&= \langle \sum_n e^{in(\phi - \mathrm{Im}\log\det(D+m+\mu\gamma_0)} \rangle_{N_f}
\end{aligned}$$

The phase density therefore follows from the moments of the phase factor.

$$\langle e^{2in\theta} \rangle_{N_f} = \frac{1}{Z_{N_f}} \left\langle \frac{\det^{n+N_f}(D + m + \mu\gamma_0)}{\det^n(D^\dagger + m + \mu\gamma_0)} \right\rangle$$

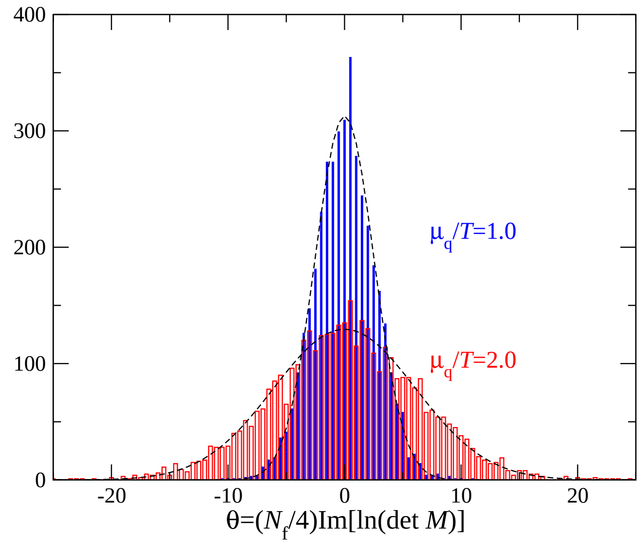
We have $2n(n + N_f)$ charged Goldstone particles. They are fermions. All uncharged Goldstone particles are bosons. We thus find

$$\langle e^{2in\theta} \rangle_{N_f} = e^{n(n+N_f)\underbrace{[G_0(\mu = 0) - G_0(\mu)]}_{-\Delta G})}$$

Phase Density

By Poisson resummation we obtain

$$\rho(\phi) = \sum_n e^{in\phi} e^{-n(n+N_f)\Delta G} = \frac{e^{\frac{1}{4}N_f^2 \Delta G}}{\sqrt{\pi \Delta G}} e^{iN_f\phi - \frac{\phi^2}{\Delta G}}.$$



Phase density in lattice QCD.

Ejiri-2007

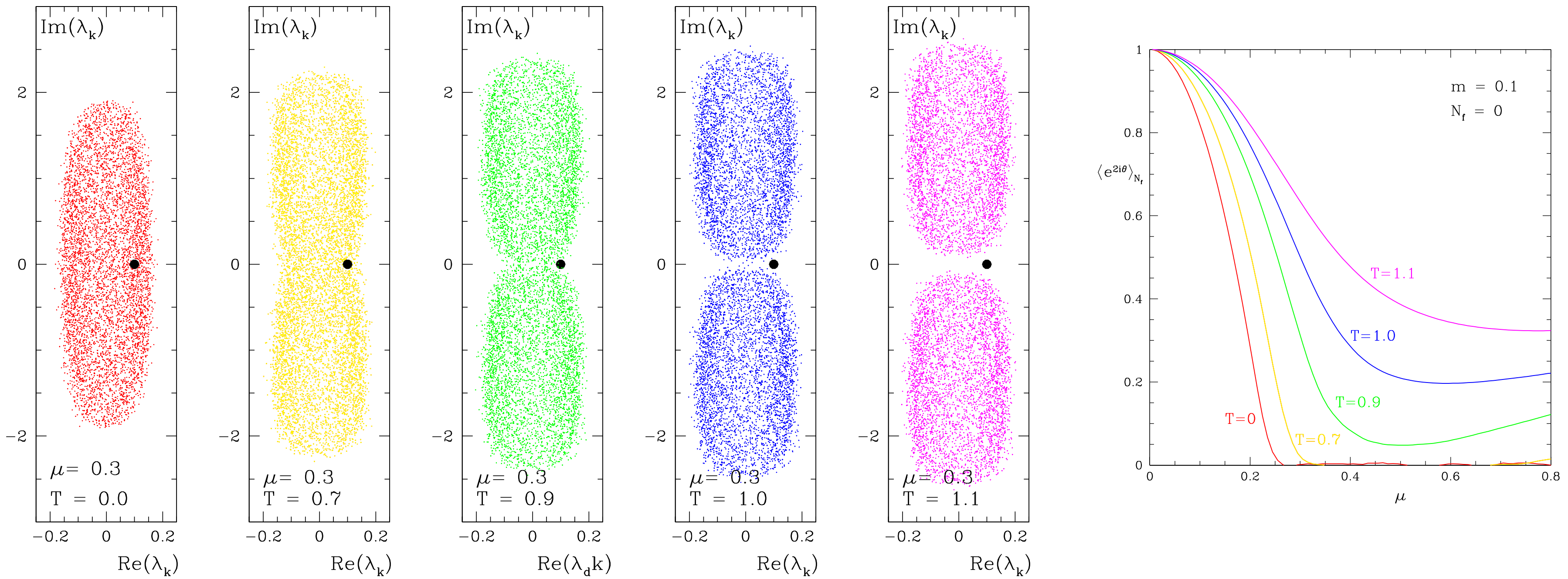
- Gaussian distribution modified by a phase.
- $\Delta G \sim VT^2\mu^2$.
- Agrees (up to the overall phase) with lattice results by Ejiri obtained by Tailor expansion of the phase angle.
- Eijiri's method to deal with the sign problem works for teflon plated observables.

V. Phase Diagram of Average Phase Factor

T -dependence of Phase Factor

Phase Diagram

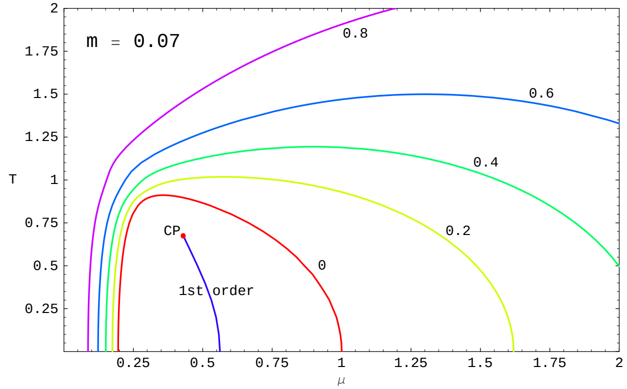
Temperature Dependence of $\langle e^{2i\theta} \rangle$



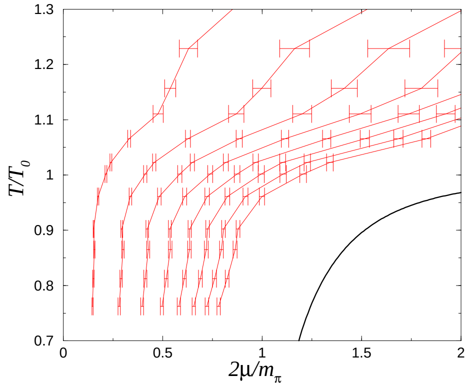
Scatter plot of Dirac eigenvalues obtained from a schematic chiral random matrix model. This random matrix model has the spectral flow of QCD and is equivalent to the zero momentum limit of a chiral Lagrangian.

The average phase factor becomes nonzero when the quark mass is outside the spectral support. The quark mass is indicated by the black dot. Ravagli-JV-2007

Ingredients for Phase Diagram of the Average Phase Factor



Analytical random matrix result for phase diagram of average phase factor. Curves show contours of equal average phase factor. Han-Stephanov-2008



Lattice results showing contour lines with equal variance of the phase of the fermion determinant.

Allton-et al-2005, Splittorff-2006

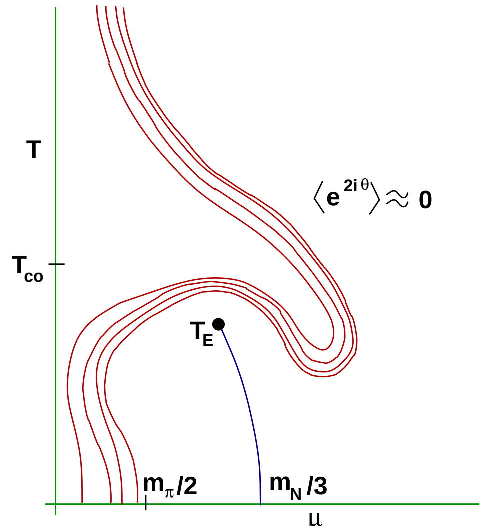
- Lattice simulations are feasible around $T_{\rm co}$ and small chemical potential.
  Fodor-Katz-2002, Allton-et al-2002, d'Elia-Lombardo-2002,
  De Forcrand-Philipsen-2002,Gavai-Gupta-2003

- Weak coupling result of QCD valid for high temperatures

$$F_{\rm QCD}(\mu,T) - F_{|{\rm QCD}|}(\mu,T) \sim \alpha_s^2\mu^2T^2.$$

Ipp-Rebhan-2003, Vuorinen-2003

Phase Diagram of the Average Phase Factor



Schematic "Phase diagram" of average phase factor at finite volume. Contour lines are curves with equal average phase factor.

There is a target of opportunity starting from $T_{\rm co}$ to the region just above the critical end point.

Interesting physics requires lattice methods that can deal with the sign problem.

Teflon plated observalbles may be evaluated in regions with a sign problems, e.g. for $T > T_c$ (see talk by Maria-Paula Lombardo).

VI. Conclusions

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- In terms of Dirac eigenvalues, a possible scenario is that the first order line is associated with the disappearance of the osciallatory region in the the Dirac spectrum.
- The sign problem is severe when the quark mass is inside the support of the Dirac eigenvalues (i.e. for $\mu > m_\pi/2)$.

Conclusions – continued

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- The width of this distribution behaves as $\sim \mu T\sqrt{V}$
- For $T < F_\pi$, the sign problem becomes manageable in the microscopic domain of QCD ($\mu^2 F_\pi^2 V \sim O(1)$).

IV. Quenched Average Phase Factor and Analyticity in μ

Quenched RMT result

Phase Factor at Imaginary Chemical Potential

Quenched Average Phase Factor

- The *quenched* average phase factor is given by

$$\langle e^{2i\theta}\rangle_{\rm q} = \left\langle \frac{\prod_k(\lambda_k + m)}{\prod_k(\lambda_k^* + m)} \right\rangle_{\rm q}.$$

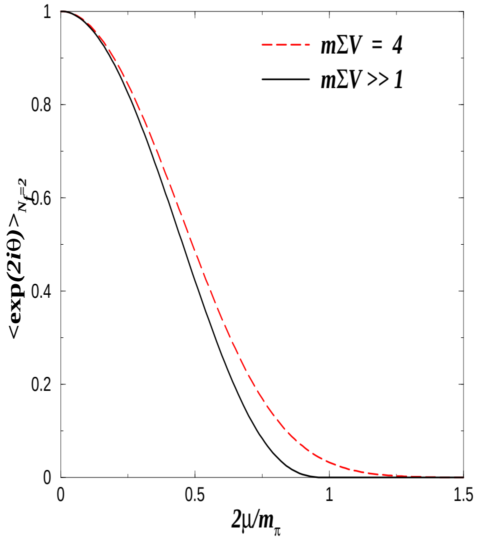
- This expression contains integrable poles.
- Is the quenched average phase factor analytic in μ ?
- We can answer this question in the microscopic domain of QCD where the QCD partition function is given by chiral random matrix theory.
- Using a version of the random matrix model proposed by Osborn (2004) the model is analytically solvable in terms of complex orthogonal polynomials.

Quenched RMT Result

$$\hat{m} = mV\Sigma$$
$$\hat{\mu} = \mu - F_\pi\sqrt{V}$$

$$\langle e^{2i\theta}\rangle_{N_f=0} = 1 - 4\hat{\mu}^2 I_0(\hat{m})K_0(\hat{m})$$
$$-e^{-2\hat{\mu}^2}\frac{1}{4\hat{\mu}^2}e^{-\frac{\hat{m}^2}{8\hat{\mu}^2}}\int_{\hat{m}}^{\infty} dx x \exp[-\frac{x^2}{4\hat{\mu}^2}]K_0\left(\frac{x\hat{m}}{4\hat{\mu}^2}\right)(I_0(x)\hat{m}I_1(\hat{m}) - xI_1(x)I_0(\hat{m}))\,,$$

Splittorff-JV-2007



Splittorff-JV-2006

- Reduces to mean field result for N_f flavors,

$$\left(1 - \frac{4\mu^2}{m_\pi^2}\right)^{N_f+1}, \quad \mu < m_\pi/2,$$

for $\hat{\mu} \to \infty,\ \hat{m} \to \infty$: and is exponentially suppressed for $\mu > m_\pi/2$.

- This expression has an essential singularity at $\mu = 0$.

- What about analytical continuation to imaginary chemical potential?

Average Phase Factor at Imaginary Chemical Potential

Analytical continuation of phase factor (Splittorff-Svetitsky-2007)

($\det^*(D+m+mu\gamma_0) = \det(D+m-\mu\gamma_0)$)

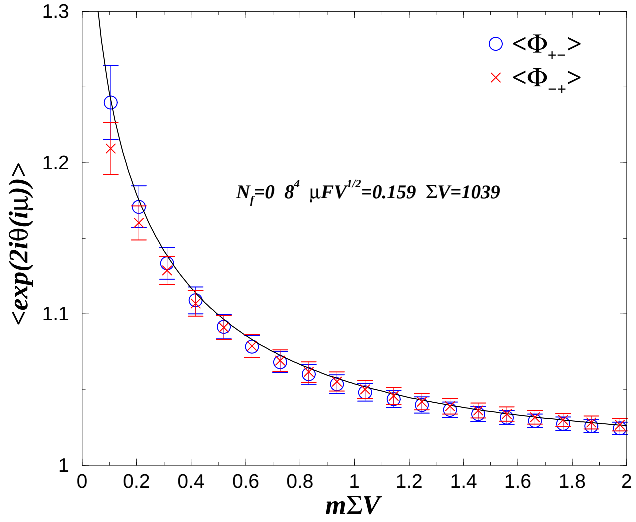
$$\left\langle \frac{\det(D+m+i\mu\gamma_0)}{\det(D+m-i\mu\gamma_0)} \right\rangle$$

Has been evaluated analytically in the microscopic domain of QCD. In the quenched case we find

$$1 - 4\hat{\mu}^2 I_0(\hat{m}) K_0(\hat{m}).$$

Damgaard-Splittorff-2006

Splittorff-JV-2006



“Phase” of the fermion determinant for imaginary chemical potential.

Splittorff-Svetitsky-2007

Discussion of Quenched Phase Factor

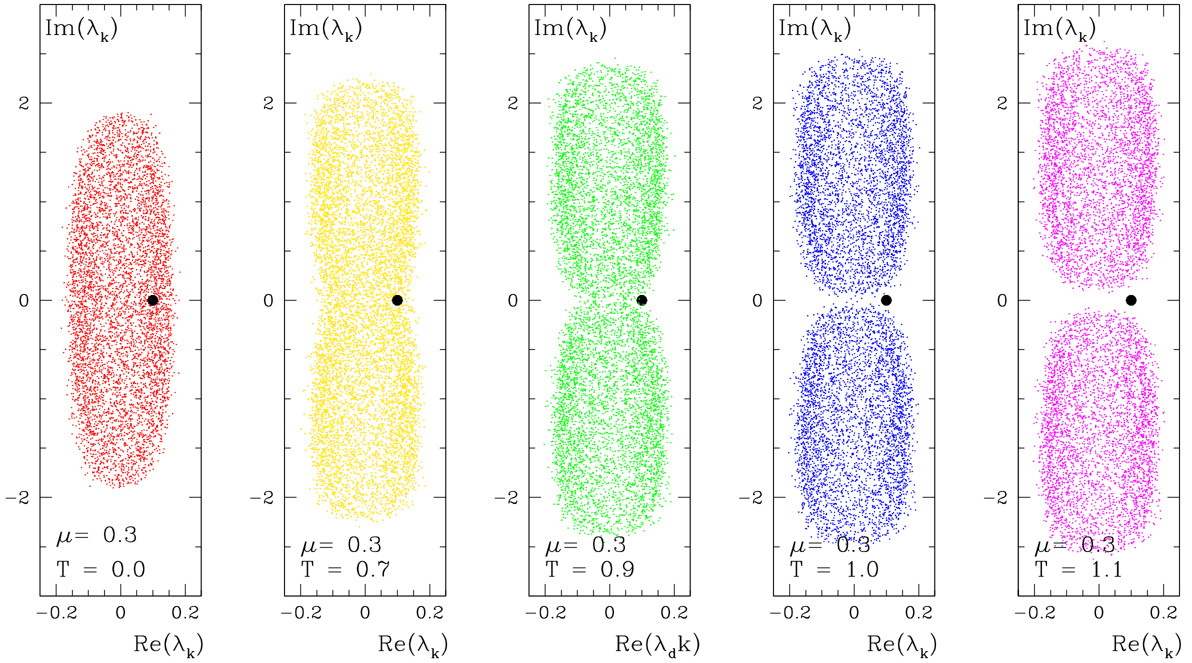
$$\langle e^{2i\theta}\rangle_{N_f=0} = 1 - 4\hat{\mu}^2 I_0(\hat{m})K_0(\hat{m})$$

Splittorff-JV-2007

$$-e^{-2\hat{\mu}^2}\frac{1}{4\hat{\mu}^2}e^{-\frac{\hat{m}^2}{8\hat{\mu}^2}}\int_{\hat{m}}^{\infty} dx x \exp[-\frac{x^2}{4\hat{\mu}^2}]K_0\left(\frac{x\hat{m}}{4\hat{\mu}^2}\right)(I_0(x)\hat{m}I_1(\hat{m}) - xI_1(x)I_0(\hat{m})),$$

- √ The first two terms can be obtained by analytical continuation from imaginary chemical potential.
- √ The second term has an essential singularity at $\mu = 0$ and cannot be obtained by analytical continuation.
- √ The second term nullifies the first term for $\mu > m_\pi/2$.
- √ The quenched average phase factor is also nonanalytic for QCD in 1d.
- √ The question is why the average phase factor is nonanalytic, and whether this should be a warning sign for other observables.

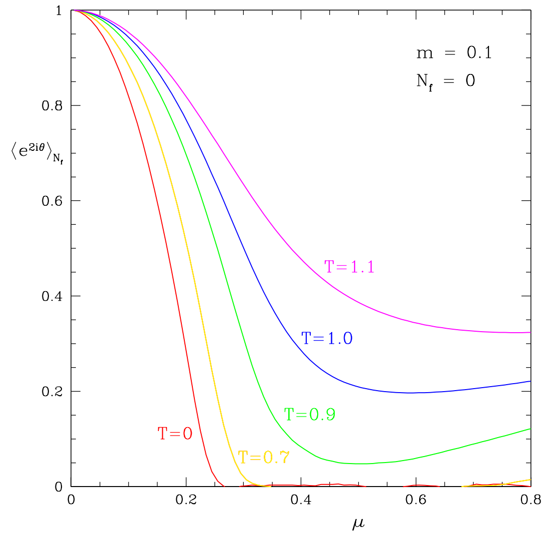
Temperature Dependence of $\langle e^{2i\theta} \rangle$



Scatter plot of Dirac eigenvalues obtained from a schematic chiral random matrix model. This random matrix model has the spectral flow of QCD and is equivalent to the zero momentum limit of a chiral Lagrangian.

The average phase factor becomes nonzero when the quark mass is outside the spectral support. The quark mass is indicated by the black dot.

Ravagli-JV-2007



Spectral Density for $N_f = 1$

The spectral density can be decomposed as

$$\hat{\rho}_{N_f=1}(\hat{x},\hat{y},\hat{m};\hat{\mu}) = \hat{\rho}_Q(\hat{x},\hat{y};\hat{\mu}) + \hat{\rho}_R(\hat{x},\hat{y},\hat{m};\hat{\mu}),$$

with ($\hat{z} = \hat{x} + i\hat{y}$)

$$\begin{aligned}\hat{\rho}_R(\hat{x},\hat{y},\hat{m};\hat{\mu}) \quad &= \quad \frac{|\hat{z}|^2}{2\pi\hat{\mu}^2} e^{-(\hat{z}^2+\hat{z}^{*\,2})/(8\hat{\mu}^2)} \\ &\times K_0(\frac{|\hat{z}|^2}{4\hat{\mu}^2}) \frac{I_0(\hat{z})}{I_0(\hat{m})} \int_0^1 dt\, t e^{-2\hat{\mu}^2 t^2} I_0(\hat{z}^* t) I_0(\hat{m}t).\end{aligned}$$

Quenched spectral density

$$\hat{\rho}_Q(\hat{x},\hat{y};\hat{\mu}) = \hat{\rho}_U(\hat{x},\hat{y},\hat{x}+i\hat{y};\hat{\mu}).$$

Osborn-2004